Special topics on set theory

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1 Weeks 5-7. Product forcing and cardinal arithmetic.

1. Let $\mathbb{P}$ be such that for every $p \in \mathbb{P}$ there exist incompatible $q \leq p$ and $r \leq p$. Show that $G \subseteq \mathbb{P}$, then $G \times G$ is not generic on $\mathbb{P} \times \mathbb{P}$.

2. Give one example of posets $\mathbb{P}$, $\mathbb{Q}$ both with the countable chain condition, such that $\mathbb{P} \times \mathbb{Q}$ has not the ccc.

3. Prove that if $\mathbb{P}$ and $\mathbb{Q}$ have the property K, then $\mathbb{P} \times \mathbb{Q}$ has the countable chain condition. A poset $\mathbb{P}$ has the property K if every uncountable set of conditions has an uncountable subset of pairwise compatible conditions.

4. The singular cardinal hypothesis SCH holds in Easton's model.

5. If κ is a singular cardinal, then there is no normal ideal on κ which contains all bounded subsets of κ.

6. Assume that κ is a regular cardinal and $\mathcal{I}$ is an ideal on κ. We call a function $g \in \kappa^\kappa$ *minimal for* $\mathcal{I}$ if and only if the following conditions:

   - $g \leq_{\mathcal{I}} \mathrm{id}_\kappa$.
   - For every $\eta < \kappa$ we have $\{\xi < \kappa : g(\xi) = \eta\} \in \mathcal{I}$.
   - If $f \in \kappa^\kappa$ is regressive on κ, then $f \circ g$ is constant on an $\mathcal{I}$-positive set.

   Prove that if $\mathcal{I}$ is σ-complete ideal on κ with $\cup\mathcal{I} = \kappa$, then there is a function $g \in \kappa^\kappa$ which is minimal for $\mathcal{I}$.

7. Assume that κ is an uncountable regular cardinal and $\Phi \in \mathrm{ON}^\kappa$ is a function. Prove that if $\sigma < \kappa$ and $S = \{\xi < \kappa : \Phi(\xi) \leq \xi + \alpha\}$ is stationary in κ, then $\| \Phi \|_{\mathcal{I}_{NS}} \leq \kappa + \alpha$. Here NS is the non-stationary ideal on κ.

8. Assume that κ, λ are cardinals satisfying $\omega \leq \lambda \leq \kappa$. Prove that there is a set $\mathcal{F} \subseteq \kappa^\lambda$ of almost disjoint functions such that $|\mathcal{F}| > \lambda$.

9. Assume that $\aleph_\eta$ is a κ-strong singular cardinal, where $\kappa = \mathrm{cf}(\aleph_\eta) > \aleph_0$. Further, let $(\eta(\xi) : \xi < \kappa)$ be a normal sequence cofinal in η such that the set $S = \{\xi < \kappa : \gimel(\aleph_{\eta(\xi)}) = \aleph_{\eta(\xi)}^+\}$ is stationary in κ. Prove that $\gimel(\aleph_\eta) = \aleph_\eta^+$.

   Hint: Prove that the set $S^* = \{\xi < \kappa : \aleph_{\eta(\xi)}$ is κ strong and $\gimel(\aleph_{\eta(\xi)}) = \aleph_{\eta(\xi)}^+\}$ is stationary and use the rules of cardinal arithmetic to prove that $\aleph_{\eta(\xi)}^\kappa = \gimel(\aleph_{\eta(\xi)})$. Finally, apply Galvin-Hajnal lemma.

   Lemma 1 (Galvin-Hajnal lemma)**.** Assume that $\aleph_\eta$ is a κ-strong singular cardinal, where $\kappa = \mathrm{cf}(\aleph_\eta) > \omega$, $(\eta_\xi : \xi < \kappa)$ is a normal sequence cofinal in η, and $\lambda > 1$ is a cardinal. Further, let $\Phi \in \mathrm{ON}^\kappa$ be an ordinal function satisfying:

   $$\aleph_{\eta(\xi)}^\lambda = \aleph_{\eta(\xi)+\Phi(\xi)}$$

   for all $\xi < \kappa$. Then:

   $$\aleph_\eta^\lambda \leq \aleph_{\eta+\|\Phi\|_{\mathcal{I}_{NS}}}$$

10. If the set $\{\xi < \omega_1 : 2^{\aleph_\xi} \leq \aleph_{\xi+\xi+2}\}$ is stationary in ω_1, then $2^{\aleph_{\omega_1}} \leq \aleph_{\omega_1+\omega_1+2}$.

11. If $\aleph_{\omega_1}$ is a strong limit cardinal, and if $\{\xi < \omega_1 : \aleph_\xi^{\aleph_0} \leq \aleph_{\xi+\xi}\}$ is a club in ω_1. Then $2^{\aleph_{\omega_1}} < \aleph_{\omega_1+\omega_1}$.