# Special topics on set theory
# Exercises weeks 3-4

Diana Carolina Montoya

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## 1 Weeks 3 and 4: Forcing

1.

   **Definition 1.** A family of sets $\mathcal{A}$ forms a $\Delta$-system with root $R$ if and only if $X \cap Y = R$, whenever $X, Y \in \mathcal{A}$ with $X \neq Y$.

   Prove the generalized version of the $\Delta$-system lemma: Let $\kappa$ be an uncountable such that $\kappa^{<\kappa}$. Let $\mathcal{A}$ be a collection of sets of cardinality less than $\kappa$ such that $|\mathcal{A}|= \kappa^+$. Then there exists a collection $\mathcal{B} \subseteq \mathcal{A}$ such that $|\mathcal{A}|= \kappa^+$ and a set $A$ such that $\mathcal{B}$ forms a $\Delta$-system with root $A$.

2. Use the item above to prove that the poset $\mathrm{Fn}_\lambda(I, J)$ has the $(|J|^{<\lambda})^+$-cc. In particular, $\mathrm{Fn}_\lambda(I, J)$ has the $(2^{<\lambda})^+$-cc whenever $|J|\leq 2^{<\lambda}$.

3. Show that in the definition of $G$ being a generic filter one can change the fact that $G$ intersects all dense sets $D$ in $M$ by the following properties, specifically:

   - A filter $G$ on $\mathbb{P}$ is generic over $M$ if and only if for every $p \in G$, if $D \in M$ is dense below $p$, $G \cap D \neq \emptyset$. $D$ is dense below $p$, if for all $q \leq p$, there exist $r \in D$ and $r \leq q$.
   - A filter $G$ on $\mathbb{P}$ is generic over $M$ if and only if for every $D \in M$ open dense set, $G \cap D \neq \emptyset$. $D$ is open dense, if it is dense and additionally if $p \in D$ and $q \leq p$ then $q \in D$.
   - A filter $G$ on $\mathbb{P}$ is generic over $M$ if and only if for every $D \in M$ predense, $G \cap D \neq \emptyset$. $D$ is predense, if every $p \in \mathbb{P}$ is compatible with some $q \in D$.
   - A filter $G$ on $\mathbb{P}$ is generic over $M$ if and only if for every $A \in M$ is a maximal antichain, $G \cap A \neq \emptyset$.

4. Assume that $\mathbb{P}, J \in M$ and in $M$, $\mathbb{P}$ is countable and $J$ is a set of size $\aleph_1$. Let $G$ be $\mathbb{P}$-generic over $M$. In $M[G]$, let $E$ be a an uncountable subset of $J$. Prove that there is an $E' \in M$ such that $E' \subseteq E$ and $E'$ is uncountable in $M$. Also give a counter-example to the existence of such an $E'$ when $\mathbb{P} = \mathrm{Fn}(J, 2)$.

   **Hint:** $E' = \{j \in J : p \Vdash j \in \dot{E}\}$ for some $p \in \mathbb{P}$, where $\dot{E}$ is a name for $E$.

5. In $M$, let $\mathbb{P} = \mathrm{Fn}(\kappa, \lambda)$ where $\aleph_0 \leq \kappa < \lambda$. Then $\lambda$ is countable in $M[G]$ and all cardinals of $M$ above $\lambda$ remain cardinals in $M[G]$. Also prove that $M[G] \models$ GCH, assuming that $M \models$ GCH.

6. Assume that $M \models \neg$ CH and let $\mathbb{P} = (\mathrm{Fn}_{\aleph_1}(I, 2))^M$ where $(|I| \geq \aleph_0)^M$. Then $M[G] \models$ CH and all cardinals $\kappa$ of $M$ with $(\aleph_1 < \kappa \leq \mathfrak{c})^M$ cease to be cardinals in $M[G]$.

   **Hint:** There is a complete embedding $\mathrm{Fn}_{\aleph_1}(\omega_1, 2)$ into $\mathbb{P}$ and $\mathrm{Fn}_{\aleph_1}(\omega_1, 2) \simeq \mathrm{Fn}_{\aleph_1}(\omega_1, \mathfrak{c})$.

7. Assume in $M$ that $\lambda$ is a singular cardinal, $|I| \geq \lambda$, $|J| \geq 2$ and $\mathbb{P} = \mathrm{Fn}_\lambda(I, J)$. Then $\lambda$ is not a cardinal in $M[G]$.