PS introduction to mathematical logic

Exercises week 6

November 10, 2016

1. Let Σ_1 and Σ_2 be sets of sentences (not necessarily finite) such that there is no model $\mathcal{M}$ such that both $\mathcal{M} \models \Sigma_1$ and $\mathcal{M} \models \Sigma_2$. Prove that there exists a sentence φ such that: Every model of Σ_1 satisfies φ and every model of Σ_2 satisfies $\neg\varphi$.

2. Assume that Γ is a theory satisfying the following:

   (a) Γ is a Henkin theory.

   (b) For any two constants c, d either $\Gamma \vdash c = d$ or $\Gamma \vdash c \neq d$ (i.e. $\Gamma \vdash \neg(c = d)$).

   (c) There are two constants a, b such that $\Gamma \vdash a \neq b$.

   Show that Γ is a complete theory.
   (Hint: For any sentence φ, consider the sentence:
   $$\exists x[(\varphi \wedge x = a) \vee (\neg\varphi \wedge x = b)]$$
   and apply that Γ is Henkin.)

3. Let I be a nonempty set, $\mathcal{U}$ an ultrafilter on I, and J an element of $\mathcal{U}$. Define $\mathcal{V}$ to be the set of $X \subseteq J$ such that $X \in \mathcal{U}$.

   (a) Show that $\mathcal{V}$ is an ultrafilter on I.

   (b) Show that if $(\mathcal{A}_i : i \in I)$ is a family of $\mathcal{L}$-structures, then $\prod_{\mathcal{U}}(\mathcal{A}_i : i \in I)$ is isomorphic to $\prod_{\mathcal{V}}(\mathcal{A}_i : i \in I)$.

4. Let $\mathcal{L}$ be the first order language whose only non-logical symbol is the binary predicate symbol $<$. Let $\mathcal{A} = (\mathbb{N}, <)$ and let $\mathcal{B} = \mathcal{A}^I/\mathcal{U}$ be the ultrapower of $\mathcal{A}$ where I is a countably infinite and $\mathcal{U}$ is a non principal ultrafilter on I.

   (a) Show that $\mathcal{B}$ is a linear order.

   (b) Show that the range of the diagonal embedding of $\mathcal{A}$ into $\mathcal{B}$ is a proper initial segment of $\mathcal{B}$. Give an explicit description of a element of B that is not in the range of this embedding.

(c) Show that $\mathcal{B}$ is not a well-ordering: that is, describe an infinite decreasing sequence of elements in $\mathcal{B}$.

Remember the following definition:

Definition 1. *Let I be an index set and $\mathcal{U}$ be an ultrafilter on I. Fix a first order language $\mathcal{L}$ and an $\mathcal{L}$-structure A. Consider the ultrapower $\mathcal{A}^I/\mathcal{U}$ of $\mathcal{A}$. Define a function δ on A by setting $\delta(a) = g_a/\mathcal{U}$, where g_a is the constant function with $g_a(i) = a$ for all $i \in I$. Then δ is an elementary embedding from A into $\mathcal{A}^I/\mathcal{U}$. (This is called the diagonal embedding; often one identifies a with $\delta(a)$ for each $a \in A$ and thereby regards A as an elementary substructure of $\mathcal{A}^I/\mathcal{U}$.)*

5. Let $\mathcal{A}$ be any $\mathcal{L}$-structure. Show that $\mathcal{A}$ can be embedded in some ultraproduct of a family of finitely generated substructures of $\mathcal{A}$.

6. Let $(I, \leq)$ be a linearly ordered set. For each $i \in I$ let $\mathcal{A}_i$ be an $\mathcal{L}$-structure, and suppose this indexed family of structures is a chain. That is, for each $i, j \in I$, we suppose $i \leq j \to \mathcal{A}_i \subseteq \mathcal{A}_j$. Prove that:

   (a) There is a well defined structure whose universe is the union of the sets A_i and which is an extension of each $\mathcal{A}_i$; moreover, such a structure is unique.

   (b) If, in addition, $\mathcal{A}_i \preceq \mathcal{A}_j$ holds whenever $i, j \in I$ and $i \leq j$, then the union of this chain of structures is an elementary extension of each $\mathcal{A}_i$. (In this situation we refer to $(\mathcal{A}_i | i \in I)$ as an elementary chain of $\mathcal{L}$-structures