Special topics in set theory

Exercises week 1-2

October 2022

1 Ordinals and cardinal arithmetic

1. Let γ be an ordinal. Show that the following are equivalent:

   (a) $\forall \alpha, \beta < \gamma$, $\alpha + \beta < \gamma$.

   (b) $\forall \alpha < \gamma$, $\alpha + \gamma = \gamma$.

   (c) $\forall X \subseteq \gamma$, $\text{type}(X) = \gamma \vee \text{type}(\gamma \setminus X) = \gamma$.

   (d) $\exists \delta$ such that $\gamma = \omega^{\delta}$.

   Recall that for a well-ordered set X, type(X) is the unique ordinal α such that $X \simeq \alpha$. Such γ is called indecomposable. The δ in (d) might equal γ. The least γ such that $\gamma = \omega^{\gamma}$ is called ϵ_0.

2. Show that if $\alpha > 0$ is an ordinal, then there are unique positive natural numbers k and $c_1, c_2, \ldots, c_k$ and ordinals $0 \leq \beta_1 < \beta_2 < \ldots < \beta_k \leq \alpha$ such that:

$$\alpha = \omega^{\beta_k} \cdot c_k + \ldots + \omega^{\beta_1} \cdot c_1$$

   This representation is called the *Cantor normal form.*

3. Prove that $(\beth_\omega)^{\aleph_0} = \Pi_{n\in\omega}\beth_n = \beth_{\omega+1}$. Here $\beth$ is the Beth function.

4. Let W be a vector space over some field F, and let $W^* = \text{Hom}(W, F)$ be the dual vector space. Consider W be a subspace of W^{**} in the usual way (identify $x \in W$ with $\varphi \to \varphi(x)$ in W^{**}). Let $W_0 = W$ and $W_{n+1} = (W_n)^{**}$, so that $W_0 \subseteq W_1 \subseteq \ldots$. Let $W_\omega = \bigcup_n W_n$. Now assume that $|F| < \beth_\omega$ and $\aleph_0 \leq \dim(W) < \beth_\omega$. Prove that $|W_\omega| = \dim(W_\omega) = \beth_\omega$.

5. Assume CH but no GCH, prove that $(\aleph_n)^{\aleph_0} = \aleph_n$ whenever $1 \leq n < \omega$.

6. Assume that α is infinite, show that $\alpha + 1$ is not a cardinal.

7. Assume that κ is an infinite cardinal and $\prec$ is the lexicographic order on $\{0, 1\}^{\kappa}$. Prove that there is no strictly increasing or strictly decreasing sequence of length κ^+.

8. Prove that if $2^{\aleph_0} > \aleph_\omega$, then $\aleph_\omega^{\aleph_0} = 2^{\aleph_0}$.

9. Assume that λ is a singular cardinal, $\kappa = \mathrm{cf}(\lambda)$ and $f \in \kappa^\kappa$. Further, let $(\mu_\xi : \xi < \kappa)$ be a sequence with supremum λ such that the set $\{\xi < \kappa : 2^{\mu_\xi} \leq \mu_\xi^{+f(\xi)}\}$ [1] is unbounded in κ. Prove that λ is a strong limit cardinal, meaning that for all $\alpha < \lambda$, then $2^\alpha < \lambda$.

10. Assume that κ and λ are cardinals with $\kappa \geq 2$ and $\lambda \geq \omega$, such that $\mathrm{cf}(\kappa^{<\lambda}) > \lambda$. Prove that the continuum function for κ is eventually constant below λ, i.e. there exists $\beta < \kappa$ such that for all $\alpha \geq \beta$, $2^\alpha = \mu$ for some $\mu < \lambda$.

[1] Here $\mu_\xi^{+f(\xi)}$ means the $f(\xi)$-th successor of μ_ξ