Some Cardinal Invariants of the Generalized Baire Spaces

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Theorem (Cantor)

For every set X, $|X| < |\mathcal{P}(X)|$, where $\mathcal{P}(X)$ is the set of all possible subsets of X.

Continuum Hypothesis(CH): $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})| = 2^{|\mathbb{N}|}$ is equal to the first uncountable cardinal $\aleph_1$.

Answer: CH is independent of the axioms of ZFC.

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(Assuming Con(ZFC)). CH is consistent. Specifically there is a model of ZFC *where* CH *holds.*

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Let M be a transitive model of ZFC (the ground model). In M, let us consider a nonempty partially ordered set $(\mathbb{P}, <)$. We call $\mathbb{P}$ a notion of forcing and the elements of $\mathbb{P}$ forcing conditions. For p, $q \in \mathbb{P}$ we say that p is stronger than q if $p < q$. A set $D \subseteq \mathbb{P}$ is dense if for every $p \in \mathbb{P}$ there is $q \in D$ stronger than p.

Inside $\mathbb{P}$ we will be interested in a special kind of sets, the generic sets (we will denote them with G). Specifically they correspond to filters that intersect all dense sets in M.

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The fundamental theorem of the forcing method is the following:

Theorem

Let M be a model of ZFC *containing* $\mathbb{P}$, *and let* $G \subseteq \mathbb{P}$ *be a generic filter over* M. *Then there is a model* $M[G]$ *of* ZFC *which includes* $M \cup \{G\}$, *has the same ordinals as* M, *and which is minimal in the sense that if* W *is any model of ZFC including* $M \cup \{G\}$, *then* $M[G] \subseteq W$.

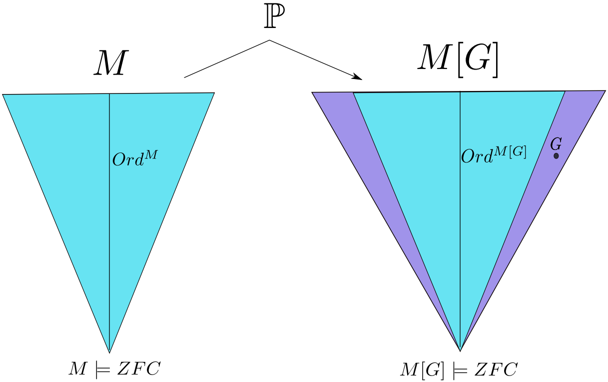

Figure 1: Forcing method

The model $M[G]$ is called a generic extension of M. The sets in $M[G]$ will be definable from G and finitely many elements of M. An important feature of the generic model is that it can be described within the ground model.

Associated with the notion of forcing $\mathbb{P}$ there is a forcing language and a forcing relation $\Vdash$, both of them are defined in the ground model M. The forcing language contains a name for every element of $M[G]$ (it is customary to denote them by dotted letters $\dot{a}$). Once we pick a generic set G, we can code the truth in $M[G]$ using the forcing relation as follows:

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Theorem

Let $\mathbb{P}$ be a forcing notion in the ground model M. If σ is a sentence of the forcing language, then for every generic $G \subseteq \mathbb{P}$ over M

$$M[G] \models \sigma \text{ if and only if } (\exists p \in G)(p \Vdash \sigma).$$

In the left-hand side σ one interprets the constants of the forcing language according to G.

Cardinal Invariants of The Continuum [2]

Cardinal invariants of the continuum are cardinals describing mostly the combinatorial or topological structure of the real line. They are usually defined in terms of ideals on the reals, or some very closely related structure such as $\mathcal{P}(\omega)/\operatorname{fin}$. Typically, they assume values between $\aleph_1$, the first uncountable cardinal, and $\mathfrak{c} = |2^\omega| = |\omega^\omega| = |\mathbb{R}|$, the cardinality of the continuum, so they are uninteresting under the continuum hypothesis $\mathfrak{c} = \aleph_1$. However, in other models of set theory, they may take different values, and they provide means for characterizing the structure of the real line in various models.

Some Examples of Cardinal Invariants

Let κ be a regular cardinal $\geq \omega$.

Definition

If f, g are functions from κ to κ, we say that $f <^* g$, if there exists an $\alpha < \kappa$ such that for all $\beta > \alpha$, $f(\beta) < g(\beta)$. In this case, we say that g eventually dominates f.

Definition

Let $\mathfrak{F}$ be a family of functions from κ to κ.

- *$\mathfrak{F}$ is dominating, if for all $g \in \kappa^\kappa$, there exists an $f \in \mathfrak{F}$ such that $g <^* f$.*
- *$\mathfrak{F}$ is unbounded, if for all $g \in \kappa^\kappa$, there exists an $f \in \mathfrak{F}$ such that $f \not<^* g$.*

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The unbounding and dominating numbers, $\mathfrak{b}(\kappa)$ and $\mathfrak{d}(\kappa)$.

Definition

- $\mathfrak{b}(\kappa) = \min\{|\mathfrak{F}| : \mathfrak{F}$ *is an unbounded family on* κ *to* $\kappa\}$
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Cardinal Invariants Associated to an Ideal

Additionally, if $\mathcal{I}$ is a σ-ideal on a set X, i.e. a nonempty family of subsets of X which is closed under subsets and countable unions. We will also demand that a σ-ideal on a set X contains all singletons of X but not X itself. Typically we consider the meager ideal $\mathcal{M}$ and the null ideal $\mathcal{N}$ respect to the standard product measure on 2^ω. The following definitions will be central to our study.

- The additivity number:

$$\mathrm{add}(\mathcal{I}) = \min\{|\mathcal{J}|\colon \mathcal{J} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{J} \notin \mathcal{I}\}.$$

- The covering number:

$$\mathrm{cov}(\mathcal{I}) = \min\{|\mathcal{J}|\colon \mathcal{J} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{J} = 2^{\omega}\}$$

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- The cofinality number:

$$\mathrm{cof}(\mathcal{I}) = \min\{|\mathcal{J}| \colon \mathcal{J} \subseteq \mathcal{I} \text{ and for all } M \in \mathcal{I} \text{ there is a } J \in \mathcal{J} \text{ with } M \subseteq J\}$$

- The uniformity number:

$$\mathrm{non}(\mathcal{I}) = \min\{|X| \colon X \subset 2^{\omega} \text{ and } X \notin \mathcal{I}\}$$

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Provable ZFC inequalities between these classical cardinal invariants in the case of the meager and null ideals on ω^ω can be summarized in Cichón's Diagram (Figure 2). An arrow between two cardinals means that the cardinal at the head of the arrow is at least as large as the cardinal at the tail.

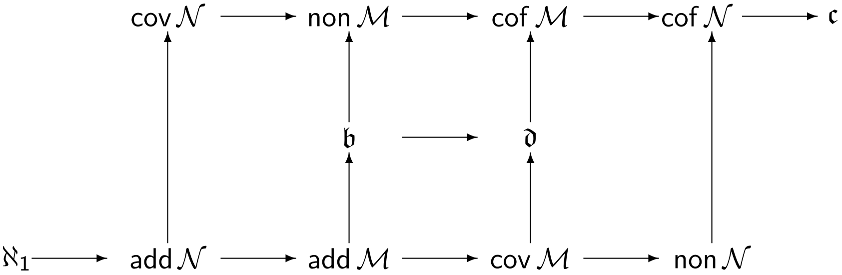

Figure 2: Cichón's diagram I

The following relations can also be established in ZFC.

Theorem (Miller and Truss.[1])

- $\operatorname{add}\mathcal{M} = \min\{\mathfrak{b}, \operatorname{cov}\mathcal{M}\}$
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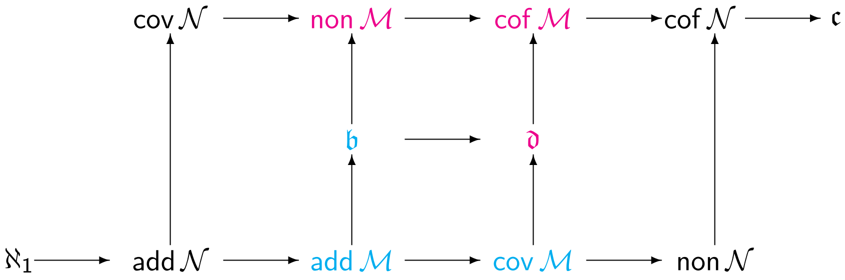

Figure 3: Cichón's diagram II

One of the main questions concerning these cardinals is whether given an assignment of values, for example $\aleph_1, \aleph_2$ it is possible to find arrangements of the diagram where this assignment holds (of course, respecting the ZFC inequalities on the diagram).

Using forcing constructions it is possible, for example, to find a model where $\mathfrak{b} = \aleph_1$ and $\mathfrak{d} = \kappa$ for any κ satisfying $\kappa^{<\aleph_0} = \kappa$.

Existing results

Cardinal invariants when $\kappa = \omega$ have been deeply studied, thus a natural question is what happens for $\kappa > \omega$. The study of this case begins with J. Cummings and S. Shelah in [4]. The main result they obtained is:

Theorem (Theorem 4 in [4])

Assume $M \models$ GCH, let $(\beta(\kappa), \delta(\kappa), \mu(\kappa))$ be an assignment of cardinals coherent with the diagram. Then there is a forcing notion $\mathbb{P}$, preserving cardinals such that in the generic extension, $\mathfrak{b}(\kappa) = \beta(\kappa)$, $\mathfrak{d}(\kappa) = \delta(\kappa)$ and $2^\kappa = \mu(\kappa)$. Moreover this can be done simultaneously for all regular cardinal κ.

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There are also results demonstrating the difference between the countable and uncountable cases. For example, the well known still open Roitman's Problem asks whether from $\mathfrak{d} = \aleph_1$ it is possible to prove that $\mathfrak{a} = \aleph_1$. In the uncountable case this has been solved in the positive.

Theorem (Theorem 2.1 in [3])

If κ is an uncountable, regular cardinal and $\mathfrak{d}(\kappa) = \kappa^+$, then $\mathfrak{a}(\kappa) = \kappa^+$

Whereas in the countable case all these cardinal invariants are at least $\aleph_1$, in the uncountable case there are some that can be smaller than κ^+ ([7]).

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Icho diagram on κ

Let $\kappa > \omega$ be a regular cardinal satisfying $\kappa^{<\kappa} = \kappa$, we are interested in the generalized Baire Space κ^κ and the cardinal invariants associated to it. As a topological space κ^κ will be endowed with the topology generated by the basic open sets $[s] = \{f \in \kappa^\kappa : s \in \kappa^{<\kappa} \text{ and } s \subseteq f\}$.

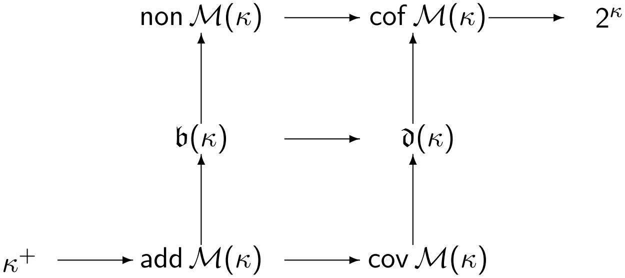

Figure 4: Icho diagram (for κ strongly inaccessible)

A general problem to study, is whether all possible constellations of the diagram are consistent (starting with the ones involving two values). For example, is it possible to find a model of ZFC where $\operatorname{cov}\mathcal{M}(\kappa) = \kappa^+$ and $\mathfrak{b}(\kappa) = \kappa^{++} = 2^\kappa$?

Our approach consists of generalizing the classical forcing notions used to control cardinal invariants in the countable case, for example: Cohen, Hechler, Mathias, Eventually Different forcing, etc. Then, in the corresponding models obtained after iterating these forcings we evaluate the values of the cardinal invariants.

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Example : κ-Cohen forcing

The natural generalization of Cohen forcing to the uncountable case is:

$$\mathbb{C}_\kappa = \{s : s \text{ is a function in } \kappa^{<\kappa}\}.$$

Let $V \models GCH$ and consider the κ-product (i.e. the product with support $< \kappa$) of length κ^{++} of the κ-Cohen forcing,

$$\mathbb{P} = \prod_{i \in \kappa^{++}} \mathbb{C}_\kappa \, .$$

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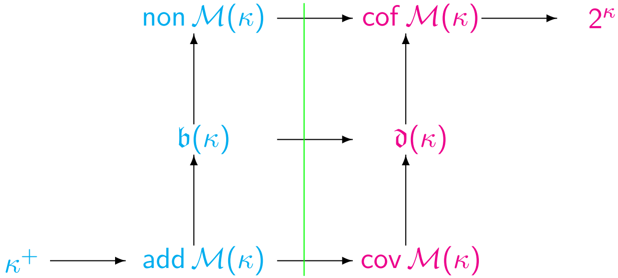

Figure 5: Effect of $\mathbb{P}$ in the diagram

κ-Mathias forcing

If we want a suitable forcing that has nice properties (κ^+-closedness for example), the natural generalization of Mathias forcing will not work. So, in this section we will assume κ is a measurable cardinal and we will take $\mathcal{U}$ to be a normal measure on it. Define the Generalized Mathias Forcing respect to the ultrafilter $\mathcal{U}$ as follows:

$$\mathbb{M}^{\kappa}_{\mathcal{U}} = \{(s, A) : s \in [\kappa]^{<\kappa} \text{ and } A \in \mathcal{U}\}$$

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With the ordering given by $(t, B) \leq (s, A)$ if and only if $t \supseteq s, B \subseteq A$ and $t \setminus s \subseteq A$.

Theorem

If $\mathcal{U}$ is a normal ultrafilter on κ, then $\mathbb{M}^{\kappa}_{\mathcal{U}}$ always adds dominating functions.

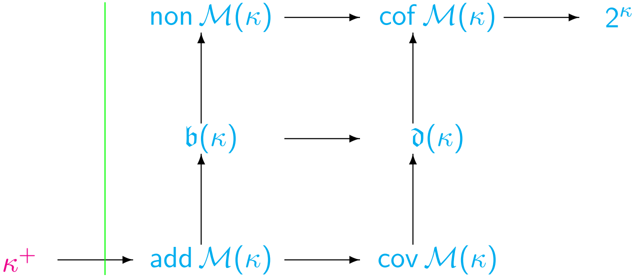

Figure 6: Effect of the $\mathbb{M}^\kappa_\mathcal{U}$ iteration of length $\lambda > \kappa^+$

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