Special topics on set theory WS2022
Tentative program

Diana Carolina Montoya *

1 Cardinal arithmetic and singular cardinals.

- Review of basic concepts- cardinality and cofinality.
- König's Theorem. Exponentiation of cardinals. GCH.
- A short review on forcing.
- Easton's theorem.

2 Arithmetic of singular cardinals.

- The singular cardinal hypothesis.
- Silver's Theorem.
- Galvin-Hajnal's theorems.

3 Large cardinals and the singular cardinals problem.

- Elementary embeddings and some large cardinal notions.
- Measurable cardinals and supercompact cardinals.
- Silver's forcing.
- Příkrý forcing.

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4 Příkrý-type forcings

- Adding many Příkrý-sequences.
- Nice systems of ultrafilters.
- Collapsing cardinals.
- Down to $\aleph_\omega$.

5 A gently introduction on pcf

Time availability dependent.

6 Introduction

This course deals with basics on cardinal arithmetic, focusing on singular cardinals. We start with an introduction to the basic concepts and results, for this purpose we introduce the following definitions:

This section was taken from Jech [Jec03].

7 Ordinals

Definition 1 (Ordinals)**.** A set x is an ordinal if it is transitive and well-ordered by $\in$. Here transitive means that every element $y \in x$ is such that $y \subseteq x$. Also, x is well-ordered by $\in$ if every non-empty subset of x has an $\in$-minimal element.

Ordinals correspond to order types of well-ordered sets, i.e. every well ordered set is isomorphic to a unique ordinal.

Definition 2 (Transfinite induction)**.** Let C be a class of ordinals and assume that:

1. if $\alpha \in C$ then $\alpha + 1 \in C$.
2. if α is a limit ordinal and for all $\beta < \alpha$, $\beta \in C$, then $\alpha \in C$.

Then C is the class of all ordinals.

Now we present the basic definitions or ordinal arithmetic:

Definition 3 (Addition of ordinals)**.**

1. $\alpha + 0 = \alpha$.
2. $\alpha + (\beta + 1) = (\alpha + \beta) + 1$.
3. $\alpha + \gamma = \sup_{\beta<\gamma} \alpha + \beta$ when γ is a limit ordinal.

Definition 4 (Multiplication of ordinals)**.**

1. $\alpha \cdot 1 = \alpha$.
2. $\alpha \cdot (\beta + 1) = (\alpha \cdot \beta) + \alpha$.
3. $\alpha \cdot \gamma = \sup_{\beta<\gamma} \alpha \cdot \beta$ when γ is a limit ordinal.

Definition 5 (Exponentiation of ordinals)**.**

1. $\alpha^1 = \alpha$.
2. $\alpha^{\beta+1} = (\alpha^\beta) \cdot \alpha$.
3. $\alpha^\gamma = \sup_{\beta<\gamma} \alpha^\beta$ when γ is a limit ordinal.

8 Cardinality

Two sets X and Y have the same cardinality (write $|X|=|Y|$) if there exists a one-to-one map of X onto Y. We define cardinality of well-ordered sets and using the axiom of choice we extend this definition to all sets.

Definition 6. And ordinal α is called a cardinal number if $|\alpha|\neq|\beta|$ for all $\beta<\alpha$.

This, if W is a well-ordered set, there is an ordinal α such that $|W|=|\alpha|$ and so:

$$|W|=\min\{\alpha : |W|=|\alpha|\}$$

Definition 7. A set X is *finite* if $|X|=|n|$ for some $n\in\mathbb{N}$. Otherwise, we say that X is *infinite*.

The ordering of cardinal number is given as follows:

$|X|\leq|Y|$ is there is a one-to-one mapping of X into Y

Also $|X|<|Y|$ if $|X|\leq|Y|$ and $|X|\neq|Y|$.

Proposition 8 (AC)**.** The order defined above is linear.

We now prove one of the most important results of Cantor.

Theorem 9. *For every set X, $|X|<|\mathcal{P}(X)|$, where $\mathcal{P}(X)$ is the* power set *of X, i.e. the set of all possible subsets of X.*

Proof. Clearly $|X|\leq|\mathcal{P}(X)|$ because the function $f : X\to\mathcal{P}(X)$, that sends every element $x\in X$ to its singleton $\{x\}$ is an injection. Now, let $g : X\to\mathcal{P}(X)$ be an arbitrary function. It is enough to show that there is a set $Y\in\mathcal{P}(X)$ such that $Y\notin\operatorname{ran}(g)$.

To this end put $Y=\{x\in X : x\notin g(x)\}$ and notice that is z is such that $g(z)=Y$, then $z\in Y$ if and only if $z\notin Y$, a contradiction. □

Theorem 10 (Cantor-Bernstein)**.** *If $|X|\leq|Y|$ and $|X|\geq|Y|$, then $|X|=|Y|$.*

Now, we define the basic operations between cardinals:

Definition 11. Let κ and λ be two infinite cardinals, we define:

- $\kappa+\lambda := |X\cup Y|$ where X and Y are disjoint sets such that $|X|=\kappa$ and $|Y|=\lambda$.
- $\kappa\cdot\lambda := |X\times Y|$ where X and Y are two sets such that $|X|=\kappa$, $|Y|=\lambda$ and $\times$ denotes the classical cartesian product of these.
- $\kappa^\lambda := |X^Y|$ where X and Y are two sets such that $|X|=\kappa$, $|Y|=\lambda$ and $X^Y=\{f : Y\to X : f \text{ is a function }\}$.

8.1 Cofinality

Definition 12 (Cofinality)**.** Let $\alpha > 0$ be a limit ordinal. We say that an increasing β-sequence $(\alpha_\xi : \xi < \beta)$, β a limit ordinal is *cofinal* in α if $\sup_{\xi<\beta} \alpha_\xi = \alpha$. Similarly, $A \subseteq \alpha$ is *cofinal* in α, if $\sup A = \alpha$. If α is an infinite limit ordinal, the cofinality of α is:

$$\begin{aligned}\mathrm{cf}(\alpha) = {} & \text{the least limit ordinal } \beta \text{ such that there is an increasing } \beta\text{-sequence } (\alpha_\xi : \xi < \beta) \\ & \text{with } sup_{\xi<\beta}\alpha_\xi = \alpha.\end{aligned}$$

Obviously for all ordinals α, $\mathrm{cf}(\alpha) \leq \alpha$.

Proposition 13. Let α be a limit ordinal, then:

1. $\mathrm{cf}(\mathrm{cf}(\alpha)) = \mathrm{cf}(\alpha)$.
2. $\mathrm{cf}(\alpha)$ is a regular cardinal.
3. If κ is an infinite cardinal, then $\kappa < \kappa^{\mathrm{cf}(\kappa)}$.

Proof.

1. Clearly $\mathrm{cf}(\mathrm{cf}(\alpha)) \leq \mathrm{cf}(\alpha)$ because $\{\beta : \beta < \alpha\}$ is cofinal in α. On the other hand, let $(\alpha_\xi : \xi < \mathrm{cf}(\alpha))$ be cofinal in α.
2. It is a consequence of the item above.
3. Let $\mathcal{F}$ be a collection of κ many functions from $\mathrm{cf}(\kappa)$ to κ. Put $\mathcal{F} = \{f_\alpha : \alpha < \kappa\}$. It is enough to show that there is a function $f : \mathrm{cf}(\kappa) \to \kappa$ such that $f \neq f_\alpha$, for all $\alpha < \kappa$. Let $\kappa = \sup_{\xi<\mathrm{cf}(\kappa)} \alpha_\xi$. Define the desired function as follows: $f(\xi) = \min\{\gamma < \kappa : \gamma \neq f_\alpha(\xi) \text{ for all } \alpha < \alpha_\xi\}$. Such γ always exists because $|\{f_\alpha(\xi) : \alpha < \alpha_\xi\}| \leq |\alpha_\xi| < \kappa$. Clearly $f \neq f_\alpha$ for all α.

□

Definition 14 (Regular and singular cardinals)**.** A cardinal number λ is called *regular*, if $\mathrm{cf}(\lambda) = \lambda$ and it is called *singular* otherwise, i.e. $\mathrm{cf}(\lambda) < \lambda$.

8.2 Cardinal arithmetic

Recall that we are interested into defining the arithmetic operations between cardinals. It turns out that addition and multiplication are both quite simple. Let κ, λ be two infinite cardinals, then:

$$\kappa + \lambda = \kappa \cdot \lambda = \max\{\kappa, \lambda\}$$

The exponentiation of cardinals is much more interesting. We will devote some time to look at this operation with more detail.

Lemma 15. If $2 \leq \kappa \leq \lambda$ and λ is infinite, then $\kappa^\lambda = 2^\lambda$.

Proof. Since $2 \leq \lambda$, then $2^\kappa \leq \lambda^\kappa$ and also $\kappa^\lambda \leq (2^\kappa)^\lambda = 2^{\kappa\cdot\lambda} = 2^\lambda$. □

If κ, λ are infinite and $\lambda < \kappa$, computing κ^λ is more difficult: On one hand, if $2^\lambda \geq \kappa$ then $\kappa^\lambda = 2^\lambda$, but if $2^\lambda < \kappa$ we can only conclude that $\kappa \leq \kappa^\lambda \leq 2^\kappa$.

Lemma 16. If $|A|= \kappa \geq \lambda$, then the set $[A]^\lambda$ has cardinality κ^λ.

Proof. $\square$

If λ is a cardinal, let:

$$\kappa^{<\lambda} = \sup\{\kappa^\mu : \mu \text{ is a cardinal and } \mu < \lambda\}$$

For the sake of completeness, we also define $\kappa^{\lambda^+} = \kappa^\lambda$ for infinite successor cardinals λ^+.

If κ is an infinite cardinal and $|A|\geq \kappa$, let

$$[A]^{<\kappa} = P_\kappa(A) = \{X \subseteq A : |X|< \kappa\}$$

8.3 Infinite sums and products

Let $\{\kappa_i : i \in I\}$ be an indexed set of cardinal numbers. We define:

$$\Sigma_{i\in I}\kappa_i = |\bigcup_{i\in I} X_i|$$

where $\{X_i : i \in I\}$ is a disjoint family of sets such that $|X_i|= \kappa_i$. This definition does not depend of the choice of the X_i's.

Also, if κ and λ are cardinals and $\kappa_i = \kappa$ for each $i < \lambda$, then:

$$\Sigma_{i<\lambda}\kappa_i = \lambda \cdot \kappa$$

Lemma 17. If λ is an infinite cardinal and $\kappa_i > 0$ for each $i < \lambda$, then:

$$\Sigma_{i<\lambda}\kappa_i = \lambda \cdot \sup_{i<\lambda} \kappa_i.$$

Proof. Let $\kappa = \sup_{i<\lambda} \kappa_i$ and $\sigma = \Sigma_{i<\lambda}\kappa_i$. First, since $\kappa_i \leq \kappa$ for all i we have $\Sigma_{i<\lambda}\kappa_i \leq \lambda \cdot \kappa$.

On the other hand, since $\kappa_i \geq 1$ for all i we have $\lambda = \Sigma_{i<\lambda}1 \leq \sigma$ and since $\sigma \geq \kappa_i$ for all i we get $\sigma \geq \sup_{i<\lambda} \kappa_i = \kappa$. Therefore $\lambda \cdot \kappa \leq \sigma$. $\square$

In particular, if $\lambda \leq \sup_{i<\lambda} \kappa_i$, we have:

$$\Sigma_{i<\lambda}\kappa_i = \sup_{i<\lambda} \kappa_i$$

Thus, we can characterize singular cardinals as follows: An infinite cardinal κ is singular just in case

$$\kappa = \Sigma_{i<\lambda}\kappa_i$$

where $\lambda < \kappa$ and for each i, $\kappa_i < \kappa$.

An infinite product of cardinals is defined using infinite products of sets. If $\{X_i : i \in I\}$ is a family of sets, then the product is defined as follows:

$$\Pi_{i\in I} X_i = \{f : f \text{ is a function on } I \text{ and } f(i) \in X_i \text{ for each } i \in I\}$$

Note that if some X_i is empty, then the product is empty. If all the X_i are non-empty, then AC implies that the product is non-empty.

If $\Pi_{i\in I}\kappa_i = |\Pi_{i\in I} X_i|$,

where $\{X_i : i \in I\}$ is a family of sets such that $|X_i|= \kappa_i$. Again, it follows from the axiom of choice that the definition does not depend on the choice of sets X_i's.

If $\kappa_i = \kappa$ for each $i \in I$, and $|I|= \lambda$, then $\Pi_{i\in I}\kappa_i = \kappa^\lambda$. Also, infinite sums and products satisfy some of the rules satisfied by finite sums and products. For instance:

- $\Pi_{i\in I}\kappa_i^\lambda = (\Pi_{i\in I}\kappa_i)^\lambda$.
- $\Pi_{i\in I}\kappa^{\lambda_i} = \kappa^{(\Sigma_i \lambda_i)}$.
- If I is a disjoint union $I = \bigcup_{j\in J} A_j$, then

$$\Pi_{i\in I}\kappa_i = \Pi_{j\in J}(\Pi_{i\in A_j}\kappa_i)$$

Infinite product of cardinals can be evaluated using the following lemma:

Lemma 18. If λ is an infinite cardinal and $(\kappa_i : i < \lambda)$ is a non-decreasing sequence of non-zero cardinals, then:

$$\Pi_{i<\lambda}\kappa_i = (\sup_i \kappa_i)^\lambda$$

Proof. Let $\kappa = \sup_{i<\lambda}$. Since $\kappa_i \leq \kappa$ for each $i < \lambda$ we have $\Pi_{i<\lambda}\kappa_i \leq \Pi_{i<\lambda}\kappa = \kappa^\lambda$.

To prove that $\kappa^\lambda \leq \Pi_{i<\lambda}\kappa_i$ we consider a partition of λ into λ-many sets of size λ, $\{A_j : j < \lambda\}$. So $\lambda = \bigcup_{j<\lambda} A_j$. Since $\Pi_{i\in A_j}\kappa_i \geq \kappa_i$ for all i, $\Pi_{i\in A_j}\kappa_i \geq \sup_{i\in A_j}\kappa_i$. Thus $\Pi_{i<\lambda}\kappa_i = \Pi_{j<\lambda}(\Pi_{i\in A_j}\kappa_i) \geq \Pi_{j<\lambda}\kappa = \kappa^\lambda$. □

Theorem 19 (König's Theorem)**.** *If $\{\kappa_i : i \in I\}$ and $\{\lambda_i : i \in I\}$ are two indexed families of cardinal numbers such that $\kappa_i < \lambda_i$, then*

$$\Sigma_{i\in I}\kappa_i < \Pi_{i\in I}\lambda_i$$

Proof. We shall show that $\Sigma_{i<\lambda}\kappa_i \ngeq \Pi_{i<\lambda}\lambda_i$: Let $(T_i : i \in I)$ be such that $|T_i|= \lambda_i$ for each $i \in I$. It suffices to show that if $(Z_i : i \in I)$ are subsets of $T = \Pi_{i\in I}T_i$ and $|Z_i|\leq \kappa_i$ for all $i \in I$, then $\bigcup_{i\in I} Z_i \neq T$.

For every $i \in I$, let S_i be the projection of Z_i into the i-th coordinate, i.e. $S_i = \{f(i) : f \in Z_i\}$. Since $|Z_i|< |T_i|$ we have that $S_i \subseteq T_i$ and $S_i \neq T_i$. Finally. let $f \in T$ be a function such that $f(i) \notin S_i$ for all $i \in I$. Clearly, $f \notin Z_i$ and so $f \notin \bigcup_{i\in I} Z_i$ as we wanted. □

Corollary 20. 1. $\kappa < 2^\kappa$ for every cardinal κ.

2. $\mathrm{cf}(2^\kappa) > \kappa$.

3. $\mathrm{cf}(\kappa^\lambda) > \lambda$.

4. $\kappa^{\mathrm{cf}(\kappa)} > \kappa$.

9 The continuum function

Recall that the generalized continuum hypothesis (GCH) states that:

$$2^{\aleph_\alpha} = \aleph_{\alpha+1} \text{ for all ordinals } \alpha$$

GCH is independent from the axioms of ZFC. Under the assumption of GCH cardinal exponentiation is evaluated as follows:

Theorem 21. *If GCH holds and κ, λ are infinite cardinals , then:*

1. *If $\kappa \leq \lambda$ then $\kappa^\lambda = \lambda^+$.*

2. *If $\mathrm{cf}(\kappa) \leq \lambda < \kappa$ then $\kappa^\lambda = \kappa^+$.*

3. *If $\lambda < \mathrm{cf}(\kappa)$ then $\kappa^\lambda = \kappa$.*

Proof. 1. Note that $\lambda^+ = 2^\lambda \leq \kappa^\lambda \leq (2^\kappa)^\lambda = 2^{\kappa \cdot \lambda} = 2^\lambda = \lambda^+$.

2. First $\kappa \leq \kappa^\lambda \leq 2^\kappa = \kappa^+$. Since $\mathrm{cf}(\kappa) \leq \lambda$ then $\kappa < \kappa^\lambda$: we have proven that $\kappa < \kappa^{\mathrm{cf}(\kappa)} \leq \kappa^\lambda$, so $\kappa^+ \leq \kappa^\lambda$.

3. $\kappa^\lambda = \sup_{\alpha<\kappa} \alpha^\lambda$ and $|\alpha^\lambda| \leq 2^{|\alpha| \cdot \lambda} = (|\alpha| \cdot \lambda)^+ \leq \kappa$.

□

We finalize with a summary of the main restrictions on the continuum function:

Theorem 22. *1. If $\kappa \leq \lambda$ then $2^\kappa \leq 2^\lambda$.*

2. $\mathrm{cf}(\kappa) > \kappa$.

3. If κ is a limit cardinal then $2^\kappa = (2^{<\kappa})^{\mathrm{cf}(\kappa)}$.

Proof. We have been already proven 1. and 2., to prove 3. let $\kappa = \Sigma_{i<\mathrm{cf}(\kappa)} \kappa_i$ where $\kappa_i < \kappa$ for all $i < \mathrm{cf}(\kappa)$. We have:

$$2^\kappa = 2^{\Sigma_{i<\mathrm{cf}(\kappa)}\kappa_i} = \Pi_{i<\mathrm{cf}(\kappa)} 2^{\kappa_i} \leq \Pi_{i<\mathrm{cf}(\kappa)} 2^{<\kappa} = (2^{<\kappa})^{\mathrm{cf}(\kappa)} \leq (2^\kappa)^{\mathrm{cf}(\kappa)} = 2^\kappa.$$

□

For regular cardinals, the conditions in the theorem above are the only restrictions on the value of the continuum function. We shall prove this later when we state Easton's theorem. For singular cardinals the situation is very different. Let's prove the following result of Silver.

Corollary 23. If κ is a singular cardinal and if the continuum function is eventually constant below κ, say with value λ, then $2^\kappa = \lambda$.

Proof. If κ is a singular cardinal that satisfies the assumptions, then there is μ such that $\mathrm{cf}(\kappa) \leq \mu < \kappa$ and $2^{<\kappa} = \lambda = 2^\mu$. Thus, $2^\kappa = (2^{<\kappa})^{\mathrm{cf}(\kappa)} = (2^\mu)^{\mathrm{cf}(\kappa)} = 2^\mu$. □

If κ is a limit cardinals and the continuum function below κ is not eventually constant, then the cardinal $2^{<\kappa} = \lambda$ is a limit of a non-decreasing sequence: $\lambda = 2^{<\kappa} = \sup_{\alpha<\kappa} 2^{|\alpha|}$ of length κ, so $\mathrm{cf}(\lambda) = \mathrm{cf}(\kappa)$. And so $2^\kappa = (2^{<\kappa})^{\mathrm{cf}(\kappa)} = \lambda^{\mathrm{cf}(\kappa)} = \lambda^{\mathrm{cf}(\lambda)}$.

If κ is regular then $\kappa = \mathrm{cf}(\kappa)$ and so $2^\kappa = \kappa^\kappa$, so $2^\kappa = \kappa^{\mathrm{cf}(\kappa)}$.

The Gimmel function $\gimel(\kappa) = \kappa^{\mathrm{cf}(\kappa)}$ determines the values of the continuum function:

Corollary 24.

1. If κ is a successor, then $2^\kappa = \gimel(\kappa)$.
2. If κ is a limit cardinal and the continuum functions is constant below κ, then $2^{<\kappa} = 2^{<\kappa} \cdot \gimel(\kappa)$.
3. Otherwise $2^\kappa = \gimel(2^{<\kappa})$.

More on singulars Can the minimum cardinal for which the continuum hypothesis fail be singular?

The negative answer to this result was given by Silver, who proved:

Theorem 25. *If κ is a singular cardinal of uncountable cofinality, and if $2^\lambda = \lambda^+$ for all $\lambda < \kappa$, then $2^\kappa = \kappa^+$.*

In the case of regular cardinals, the full set of rules for exponentiation can be determined when assuming GCH.

On the contrary, it turns out that for the arithmetic of singular cardinals, it is not just the continuum function 2^κ which determines exponentiation, but also the function $\kappa^{\mathrm{cf}(\kappa)}$.

10 Models of set theory

Recall that the language of set theory consist of one binary predicate symbol $\in$.

Definition 26. Let M be a class, E a binary relation on M and let $\varphi(x_1, \ldots, x_n)$ be a formula of the language of set theory. The *relativization* of φ to (M, E) is the formula $\varphi^{M,E}(x_1, \ldots, x_n)$ defined inductively as follows:

$$
\begin{gathered}
(x \in y)^{M,E} = xEy \\
(x = y)^{M,E} = x = y \\
(\neg\varphi)^{M,E} \leftrightarrow \neg(\varphi^{M,E}) \\
(\varphi \wedge \psi)^{M,E} \leftrightarrow \varphi^{M,E} \wedge \psi^{M,E}. \\
(\exists x \varphi)^{M,E} \leftrightarrow \exists x \in M \varphi^{M,E}.
\end{gathered}
$$

One writes $(M, E) \models \varphi(x_1, \ldots, x_n)$ and says the model (M, E) satisfies φ.

We point out that while this is a legitimate statement in every particular instance of φ, the general satisfaction relation is formally indefinable in ZF.

10.1 Relative consistency

Let T be a mathematical theory (in practice ZF or ZFC) and let A be an additional axiom. We say that $T + A$ is relative consistent to T if the following implication holds:

If T is consistent, so is $T + A$

If both A and $\neg A$ are consistent with T we say that A is independent of T.

10.2 Transitive models and Δ_0-formulas

If M is a transitive class, then the model (M, E) is called a transitive model.

Definition 27. A formula of set theory is Δ_0 if:

- it has no quantifiers, or
- it is $\varphi \wedge \psi$, $\varphi \vee \psi$, $\neg\varphi$, $\varphi \to \psi$ or $\varphi \leftrightarrow \psi$, where φ and ψ are Δ_0.
- it is $(\exists x \in y)\varphi$ or $(\forall x \in y)\varphi$ where φ is Δ_0.

Lemma 28 (Absolutness of Δ_0-formulas)**.** If M is a transitive class and φ is a Δ_0-formula, then for all $x_1, \ldots, x_n$.

$$\varphi^M(x_1, \ldots, x_n) \leftrightarrow \varphi(x_1, \ldots, x_n)$$

Warning: While many of the basic concepts in set theory can be expressed as Δ_0-formulas, the following expressions are not:

$Y = \mathcal{P}(X)$, $|Y| = |X|$, α is a cardinal, $\beta = \mathrm{cf}(\alpha)$, α is regular.

11 Forcing

The content of this section was taken from Kunen [**Ku**] Forcing is a technique used to produce models of ZFC with specific properties. It was developed by Paul Cohen and up to this day it is a main tool used in different areas within set theory. Cohen proved (using this method) that given a model $M \models$ ZFC, there is a *generic extension* $N \supseteq M$ which is also a model of ZFC and in which CH is false, i.e. $2^{\aleph_0} > \aleph_1$.

To build the model NTo build the model N, we fix some forcing poset $\mathbb{P}$ will gives us a model N that satisfies ZFC. Exactly what N will satisfy beyond ZFC depends on the $\mathbb{P}$ we choose.

Definition 29.

- A filter $G \subseteq \mathbb{P}$ is called $\mathbb{P}$-*generic over* M if and only if it meets all dense $D \subseteq \mathbb{P}$ with $D \in M$.

- $\mathbb{P}$ is *atomless* if for all $r \in \mathbb{P}$ there are $p, q \leq r$ such that $p \perp q$), then $G \notin M$.

Regarding the first item, since we only deal with countable models, such a filter will always exist. A consequence of the second item is that if $G \subseteq \mathbb{P}$ is generic, $G \notin M$. The generic extension $N \supseteq M$ will be denoted $M[G]$; N is the minimal model of ZFC containing M, G, and having the same ordinals as M.

For a simple example, let's look at Cohen forcing:

Definition 30. For any I and J, $\mathrm{Fn}(I, J)$ is the set of all finite partial functions from I to J, that is:

$$\mathrm{Fn}(I, J) = \{p : I \to J : p \text{ is the graph of a function } \wedge p \in [I \times J]^{<\omega}\}$$

Ordered by reverse inclusion, i.e. $q \leq p$ if and only if $q \supseteq p$.

Then $\mathbb{P} \in M$ since the definition of being a finite function is an absolute notion. Similarly, the following are subsets of $\mathbb{P}$ in M for all $i \in I$:

$$D_i = \{q \in \mathrm{Fn}(I, J) : i \in \mathrm{dom}(q)\}$$

Moreover, one can show these are dense subsets of $\mathbb{P}$,so that if G is $\mathbb{P}$-generic over M, $G \cap D_i \neq \emptyset$ for all $i \in I$.

By the definition of the ordering $\leq$ and the fact that G is a filter (i.e. only contains pairwise compatible elements), $f_G := \bigcup G \in M[G]$ is a function from I to J, and since G is generic we know f is total. Now, say $I = \kappa \times \omega$ and $J = 2$ and $\kappa < o(M)$, so $\kappa \in M$. By setting

$$A_\alpha = \{n \in \omega : f_G(\alpha)(n) = 1\},$$

we can think of f_G as coding κ many subsets of ω. In other words, for fixed $\alpha < \kappa$ the function $f_G(\alpha) : \omega \to 2$ acts as a characteristic function for the set A_α. Also if $\alpha \neq \beta < \kappa$, then $A_\alpha \neq A_\beta$ since G intersects each of the the following dense sets in $\mathbb{P}$:

$$\{E_{\alpha,\beta} = \{p \in \mathbb{P} : \exists n((\alpha, n), (\beta, n) \in \mathrm{dom}(p) \ \wedge \ p(\alpha, n) \neq p(\beta, n)\}$$

Recall that when we speak of the ordinals $\omega_1^M < \omega_2^M \in \mathrm{On} \cap M$, even though we living in V can see that these are countable ordinals (since M is countable), these are defined inside M to be the first and second uncountable cardinals.

Suppose $\kappa = \omega_2^M$. Then $N = M[G]$ contains the sequence $\langle A_\alpha : \alpha < \kappa\rangle$, which are κ many distinct subsets of ω, implying $|\mathcal{P}(\omega)| = 2^\omega \geq \kappa = \aleph_2^M$, and therefore

$$M[G] \models \neg CH + ZFC$$

.

If also $M \models GCH$, then one can show that in $M[G]$, $2^{\aleph_0} \leq \kappa$, meaning the continuum will be exactly size κ.

Warning! One of the important properties of this specific poset $\mathbb{P}$ is that $\omega_1^M = \omega_1^{M[G]}$ and $\kappa = \omega_2^M = \omega_2^{M[G]}$, so that to the people in $M[G]$, the sequence $\langle A_\alpha : \alpha < \kappa\rangle$ really is an ω_2 length sequence.

12 Generic Extensions

Let M be a countable transitive model of $ZF - P$. When $(\mathbb{P}, \leq, \mathbb{1})$ is a forcing poset, let $\mathbb{P} in M$ abbreviate $(\mathbb{P}, \leq, \mathbb{1}) \in M$. Note that the definition of being a forcing poset (i.e. a partial order with a maximal element) is absolute for transitive models of $ZF - P$, and in many cases absoluteness also implies that the ordering $\leq$ on $\mathbb{P}$ is an *element* of M, but this is not always the case. It is the case, however, for the poset we used above, $\mathrm{Fn}(I, J)$.

Definition 31.

- $D \subseteq \mathbb{P}$ is dense in $\mathbb{P}$ if for all $p \in \mathbb{P}$, there is $q \leq p$, $q \in D$.
- Given $G \subseteq \mathbb{P}$, we say that G is a *filter* if:
  - $G \neq \emptyset$.
  - If $p \leq q$ and $p \in G$, then $q \in G$.
  - If $p, q \in G$, there is $r \in G$ such that $r \leq p$ and $r \leq q$.

Definition 32 (Genericity)**.** For a forcing notion $\mathbb{P}$, G is $\mathbb{P}$-generic over the model M if G is a filter on $\mathbb{P}$ and $G \cap D \neq \emptyset$ for all dense $D \subseteq \mathbb{P}$ such that $D \in M$.

Lemma 33 (Generic filter existence lemma)**.** Let M be a countable transitive model of ZF $\setminus$¶ and let $\mathbb{P} \in M$ be a forcing poset. Then, for every $p \in \mathbb{P}$, there exists a filter G on $\mathbb{P}$ such that $p \in G$ and G is $\mathbb{P}$-generic over M.

Proof. Since M is countable, one can enumerate the dense subsets of $\mathbb{P}$ in M as $(D_n : n \in \omega)$. Now, construct a sequence $(p_n : n \in \omega) \subseteq \mathbb{P}$ such that $p_n \in D_n$ and $p_{n+1} \leq p_n$ for all $n \in \omega$.

Let $p_0 \in D_0$ such that $p_0 \leq p$; if we have already built p_n, use that D_{n+1} is dense to pick $p_{n+1} \in D_{n+1}$ such that $p_{n+1} \leq p_n$. Hence $G = \{q \in \mathbb{P} : \exists n (p_n \leq q)\}$ is $\mathbb{P}$-generic over M. □

Notice that the statement "being dense" is absolute, but the enumeration of the dense sets in M of order type ω must take place outside of M, so in the usual situations $G \notin M$.

Lemma 34. If $\mathbb{P} \in M$ is atomless (i.e. for all $r \in \mathbb{P}$ there exists $p, q \in \mathbb{P}$ such that p and q are incompatible) and G is $\mathbb{P}$-generics over MN, then $G \notin M$.

Proof. Let $D = \mathbb{P} \setminus G$, then D is dense and if G were to be an element of M, so would be D and $G \cap D = \emptyset$, a contradiction.

Why D is dense?: Take $p \in \mathbb{P}$, if $p \in D$ we are done, otherwise $p \in G$ and we can use that $\mathbb{P}$ is atomless to find $q, r \in \mathbb{P}$, $q, r \leq p$ so that q and r are incompatible. Then q and p cannot be simultaneously in G (elements in a filter have to be pairwise compatible), so either q or $r \in D$. □

We define now $M[G]$.

Definition 35. (Definition by recursion) τ is a $\mathbb{P}$-*name* if τ is a binary relation and for all $\langle \sigma, p \rangle \in \tau$, σ is a $\mathbb{P}$-name and $p \in \mathbb{P}$.

Denote by $V^{\mathbb{P}}$ the class of all $\mathbb{P}$ names.

For example, $\emptyset$ is a $\mathbb{P}$-name, and therefore so is $\langle \emptyset, p\rangle$ for all $p \in \mathbb{P}$ and $\{\langle\langle \emptyset, \not\Vdash\rangle, p\rangle\}$, $\{\langle \emptyset, p\rangle, \langle \emptyset, q\rangle\}$, etc...

Definition 36. If M is a transitive model of $ZF - P$, $\mathbb{P} \in M$, then $M^{\mathbb{P}} = V^{\mathbb{P}} \cap M$, i.e. the set of $\tau \in M$ such that $M \models$ "τ is a $\mathbb{P}$-name".

Definition 37. Definition by recurison If τ is a $\mathbb{P}$-name and $G \subseteq \mathbb{P}$, define

$$\mathrm{val}(\tau, G) = \tau_G = \{\mathrm{val}(\sigma, G) : \exists p \in \mathbb{P}\ \langle \sigma, p\rangle \in \tau\}$$

.

Then $M[G] = \{\tau_G : \tau \in M^{\mathbb{P}}\}$ whenever M is a transitive model of $ZF - P$ and $\mathbb{P} \in M$.

Example: Suppose we have $\mathbb{P}$-names $\sigma^1, \sigma^2, \sigma^3$. If we want to collect the things they name, we can set

$$\tau = \{\langle \sigma^1, \mathbb{1}\rangle, \langle \sigma^2, \mathbb{1}\rangle, \langle \sigma^3, \mathbb{1}\rangle\}$$

.

Then τ is a $\mathbb{P}$-name, and if $G \subseteq \mathbb{P}$ is any filter, then $\mathbb{1} \in G$, and so $\tau_G = \{\sigma^1_G, \sigma^2_G, \sigma^3_G\}$.

If however $p_1, p_2, p_3 \in \mathbb{P}$, and say $\pi = \{\langle \sigma^1, p_1\rangle, \langle \sigma^2, p_2\rangle, \langle \sigma^3, p_3\rangle\}$, then π_G is *conditional* on which $p \in \mathbb{P}$ end up in G.

But we want to make sure all elements in the ground model M end up in any generic extension, so

Definition 38. For $\mathbb{P} = (\mathbb{P}, \leq, \mathbb{1})$ a forcing poset and any set x, define, by recursion on elements in x, the *canonical name* (or "check name") of x to be the set

$$\check{x} = \{\langle \check{y}, \mathbb{1}\rangle : y \in x\}$$

.

Proposition 39. Suppose M is a transitive model of $ZF - P$, $\mathbb{P} \in M$, and $G \subseteq \mathbb{P}$ a filter. Then:

1. $\forall x \in M\ (\check{x} \in M^{\mathbb{P}} \wedge \mathrm{val}(\check{x}, G) = x)$.
2. $M \subseteq M[G]$.

Proof. It suffices to prove (1) since (2) follows immediately. If $x \in M$, then $\check{x} \in M$ by absoluteness, and by recursion we have that $\check{x}$ is indeed a $\mathbb{P}$- name. Therefore $\check{x} \in M \wedge \check{x} \in V^{\mathbb{P}}$, so $\check{x} \in M \cap V^{\mathbb{P}} = M^{\mathbb{P}}$. Similarly by recursion one shows $\mathrm{val}(\check{x}, G) = x$ since the former set is equal to $\{\mathrm{val}(\check{y}, G) : y \in x\}$. $\square$

We also want a canonical name for the filter G; note that even though in the interesting (nontrivial) cases $G \notin M$, still in M one can define a name which, in the extension by G, will always be evaluated to be that set G:

Definition 40. For a forcing poset $\mathbb{P}$, let

$$\Gamma = \{\langle \check{p}, p\rangle : p \in \mathbb{P}\}$$

.

Therefore,

Proposition 41. If M is a transitive model of $ZF - P$, $\mathbb{P} \in M$, and $G \subseteq \mathbb{P}$ is a filter, then Γ is a $\mathbb{P}$-name and $\Gamma_G = G$. Hence $G \in M[G]$.

Proof. By definition, $\Gamma_G = \{\check{p}_G : p \in G\} = \{p : p \in G\} = G$. $\square$

Let us define $\mathbb{P}$ names which we want to be evaluated in the generic extension to be the unordered pair and the ordered pair of any two elements of $M[G]$.

Definition 42. For σ, τ $\mathbb{P}$-names, let $\mathrm{up}(\sigma,\tau) = \{\langle\sigma, \mathbb{1}\rangle, \langle\tau, \mathbb{1}\rangle\}$, and $\mathrm{op}(\sigma,\tau) = \mathrm{up}(\mathrm{up}(\sigma,\sigma), \mathrm{up}(\sigma,\tau))$.

One can check

$$\mathrm{val}(\mathrm{up}(\sigma,\tau), G) = \{\sigma_G, \tau_G\}$$

and

$$\mathrm{val}(\mathrm{op}(\sigma,\tau), G) = (\sigma_G, \tau_G),$$

which is what we wanted.

Proposition 43. Under the hypotheses of the previous Proposition, $M[G]$ is transitive, and $M[G] \models$ Extensionality, Foundation, Pairing, and Union.

Proof. By definition, the elements of $M[G]$ are sets of the form τ_G for $\tau \in M^{\mathbb{P}}$, and every element of τ_G is a set of the form σ_G for $\sigma \in M^{\mathbb{P}}$; this shows that $M[G]$ is transitive.

Transitivity implies $M[G]$ satisfies the Axiom of Extensionality. Foundation is satisfied because this is true for every class.

That $\mathrm{up}(\sigma,\tau)$ is a name in $M^{\mathbb{P}}$ whenever $\sigma, \tau \in M^{\mathbb{P}}$ shows that $M[G]$ satisfies the Pairing Axiom.

To show the Union Axiom is satisfied, we show that for any $a \in M[G]$, there exists $b \in M[G]$ such that $\bigcup a \subseteq b$.

$\square$

Proposition 44. $M[G]$ is a transitive model of ZFC.

Although $M[G]$ is bigger than M, it is not too much.

Proposition 45.

1. $\mathrm{rank}(\tau_G) \leq \mathrm{rank}(\tau)$;
2. $\mathrm{On} \cap M[G] = \mathrm{On} \cap M$;
3. $|M[G]| = |M|$.

Proof.

1. $\mathrm{rank}(\tau_G) = \sup\{\mathrm{rank}(x) + 1 : x \in \tau_G\}$ by the definition of rank (see Kunen Chapter I.9). Then supposing by induction we have shown that for all $\sigma \in \mathrm{dom}(\tau)$, $\mathrm{rank}(\sigma_G) \leq \mathrm{rank}(\sigma)$ and the fact that $\mathrm{rank}(\sigma) < \mathrm{rank}(\tau)$, $\sup\{\mathrm{rank}(x) + 1 : x \in \tau_G\} \leq \sup\{\mathrm{rank}(\sigma_G) + 1 : \sigma \in \mathrm{dom}(\tau)\} \leq \mathrm{rank}(\tau)$.
2. The inclusion $\supseteq$ is immediate, so suppose α is an ordinal in $M[G]$. Then $\alpha = \tau_G$ for some $\tau \in M^{\mathbb{P}}$, and $\mathrm{rank}(\tau)$ is some ordinal in M since it is absolute, so since $\alpha = \mathrm{rank}(\alpha) \leq \mathrm{rank}(\tau)$, $\alpha \in M$.

3. $M^{\mathbb{P}} \subseteq M \subseteq M$ and $|M[G]| \leq |M^{\mathbb{P}}|$.

$\square$

We would like now to prove that $M[G]$ is a model of ZFC as well. Although in this notes we won't do that, we would like now to motivate the two main theorems of forcing.

Suppose we would like to prove $M[G] \models$ Comprehension. This is indeed, one of the hardest axioms to prove: Let's try to exemplify why:

Suppose $\varphi(x, y)$ is a formula and $\sigma \in \mathbb{P}$, we want to argue that the set $S = \{n \in \omega : \varphi(n, \sigma_G)^{M[G]}\}$ is an element of the model $M[G]$.

This is not immediate obvious even in simple cases such us when $\varphi(x, y) = x \in y$ and so $S = \omega \cap \sigma_G$. We would like to find a $\mathbb{P}$-name $\tau \in M^{\mathbb{P}}$ such that $S = \tau_G$.

The desired τ_G is in fact, as follows:

$$\{(\check{n}, p) : n \in \omega \wedge p \in \mathbb{P} \wedge p \Vdash \varphi(\check{n}, \sigma)\}$$

Now, we will define the forcing relation $\Vdash$ and we will state the main theorems of forcing, whose goal is to try to answer the following question (for our particular example): Why is $\tau_G \in M[G]$? and why is $\tau_G = S$?. We define:

Definition 46. Given a poset $\mathbb{P}$, the $\mathbb{P}$-forcing language $\mathcal{FL}_{\mathbb{P}}$ is the class of logical formulas formed using the binary relation $\in$ and that all names $V^{\mathbb{P}}$ as constants.

Definition 47. Let ψ be a sentence in $\mathcal{FL}_{\mathbb{P}} \cap \mathcal{M}$, $M[G] \models \psi$ corresponds to the usual model theoretical meaning, by interpreting $\in$ as membership and τ as τ_G.

Definition 48. Assume $\mathcal{M} \models \text{ZF} \setminus \P$, $\mathbb{P} \in M$ is a forcing notion and ψ is a sentence in $\mathcal{FL}_{\mathbb{P}} \cap \mathcal{M}$. Then:

$$p \Vdash_{\mathbb{P},M} \psi \text{ holds }^{1} \leftrightarrow M[G] \text{ for all } G \subseteq \mathbb{P} \text{ generic over } M \text{ such that } p \in G$$

Lemma 49. If $p \Vdash \varphi$ and $q \leq p$, then $q \Vdash \varphi$.

The two most important theorems of forcing are stated now:

Theorem 50 (The truth lemma). *Let M be a countable transitive model for* $\text{ZF} \setminus \P$*, $P \in \mathbb{P}$ be a forcing notion, ψ be a sentence of $\mathcal{FL}_{\mathbb{P}} \cap \mathcal{M}$ and G be $\mathbb{P}$-generic over M. Then:*

$$M[G] \models \psi \leftrightarrow \exists p \in G \textit{ such that } p \Vdash \psi$$

Theorem 51 (Definability lemma). *Let M be a countable transitive model of* $\text{ZF} \setminus \P$*. Let $\varphi(x_1, \ldots, x_n)$ be a formula in $\mathcal{L} = \{\in\}$. Then the set*

$$\{(p, \mathbb{P}, \leq, \nVdash, \nu_1, \ldots, \nu_n) : (\mathbb{P}, \leq, \nVdash) \textit{ is a forcing notion } \wedge p \in \mathbb{P} \wedge (\mathbb{P}, \leq, \nVdash) \in M \wedge \nu_1, \ldots, \nu_n \in M^{\mathbb{P}} \wedge p \Vdash_{\mathbb{P},M} \varphi(\nu_1, \ldots, \nu_n)\}$$

is definable subset of M without parameters.

[1] We read p forces ψ

13 Computing cardinal exponentiation

Let M be a countable transitive model of ZFC. We shall use posets of the form $\mathrm{Fn}(I,J)$ to get models $M[G]$ in which $2^{\aleph_0}$ is any value not contradicting König's theorem.

Example 52. Let $\mathbb{P} = \mathrm{Fn}(I,J)$ where $I,J \in M$ and I,J are both infinite. Let G be $\mathbb{P}$-generic over M and let $f = \bigcup G$, then $f \in M[G]$ and f is a function from I to J.

Let $I = \omega$ and $J = \omega_5^M$. Now $\omega^M = \omega$ but ω_5^M is a countable ordinal, although from the point of view of M this is the fifth uncountable cardinal.

$M[G]$ contains a map between ω onto ω_5^M, so $M[G] \models \omega_5^M$ is countable.

Since $M[G] \models$ ZFC, it has his own uncountable cardinals $\omega_1^{M[G]}, \omega_2^{M[G]}, \dots, \omega_5^{M[G]}$ but these are all above ω_5^M. One can prove that these are in fact $\omega_6^{M[G]}, \omega_7^{M[G]}, \dots, \omega_{10}^{M[G]}$.

For now, we concentrate in situations in which M and $M[G]$ have the same cardinals.

Definition 53. For a forcing poset $\mathbb{P} \in M$:

1. $\mathbb{P}$ preserves cardinals if and only if for all generic $G \subseteq \mathbb{P}$: $(\beta \text{ is a cardinal })^M$ if and only if $(\beta \text{ is a cardinal })^{M[G]}$ for all $\beta < o(M)$.
2. $\mathbb{P}$ preserves cofinalities if and only if for all $G \subseteq \mathbb{P}$ generic $\mathrm{cf}^M(\gamma) = \mathrm{cf}^{M[G]}(\gamma)$ for all limit $\gamma < o(M)$.

Recall that: Being a cardinal can be written as: $\forall \alpha < \beta\ \forall f\ \neg(f : \alpha \to \beta)$ (onto).

Lemma 54. For a forcing poset $\mathbb{P} \in M$.

1. $\mathbb{P}$ preserves cofinalities if and only if for all generics $G \subseteq \mathbb{P}$:
   $$(*) \text{ for all limit } \beta \text{ with } \omega < \beta < o(M)\ (\beta \text{ is regular})^M \to (\beta \text{ is regular})^{M[G]}$$
2. If $\mathbb{P}$ preserves cofinalities then it preserves cardinalities.

Proof. 1. Assume $(*)$ and fix a limit $\gamma < o(M)$, let $\beta = \mathrm{cf}^M(\gamma)$. We prove that $\mathrm{cf}^{M[G]}(\gamma)$.

Fix $X \in \mathcal{P}(\gamma) \cap M$ with $\mathrm{ot}(X) = \beta$ and $\sup(X) = \gamma$. We know that $(\beta \text{ is regular})^M$ and by hypothesis this implies that $(\beta \text{ is regular})^{M[G]}$. Thus $\mathrm{cf}^{M[G]}(\gamma) = \mathrm{cf}^{M[G]}(\beta) = \beta$.

2. M and $M[G]$ have the same regular cardinals. Now, in ZFC each cardinal is regular or the limit of regular cardinals.

□

13.1 The chain condition

Definition 55.
- Let $\mathbb{P}$ be a forcing poset. Then $p,q \in \mathbb{P}$ are compatible (write $p \parallel q$) if there is a common extension $r \in \mathbb{P}$, $r \leq p$ and $r \leq q$.
- Let $\mathbb{P}$ be a forcing poset. Then $p,q \in \mathbb{P}$ are compatible (write $p \parallel q$) if there is a common extension $r \in \mathbb{P}$, $r \leq p$ and $r \leq q$.

- A subset $A \subseteq \mathbb{P}$ is an antichain if all its elements are pairwise incompatible.

Definition 56 (The countable chain condition)**.** We say that $\mathbb{P}$ has the countable chain condition (ccc) if every antichain on $\mathbb{P}$ is countable.

Theorem 57. *If $\mathbb{P} \in M$ and ($\mathbb{P}$ is ccc $)^M$, then $\mathbb{P}$ preserves cofinalities.*

We first prove the following lemma:

Lemma 58. Assume that $\mathbb{P} \in M$, ($\mathbb{P}$ is ccc$)^M$ and $A, B \in M$. Let G be $\mathbb{P}$-generic over M. Fix $f : A \to B$, $f \in M[G]$, then there is a function $F : A \to \mathcal{P}(B)$ with $F \in M$ such that for all $a \in A$, $f(a) \in F(a)$ and $(|F(a)| \leq \aleph_0)^M$.

Proof. Fix a $\mathbb{P}$-name $\dot{f} \in M^{\mathbb{P}}$ for f such that $\dot{f}_G = f$. Then $\dot{f} : \check{A} \to \check{B}$ is a sentence in $\mathcal{FL}_{\mathbb{P}} \cap \mathcal{M}$ about the names $\dot{f}, \check{A}, \check{B}$ and the model $M[G]$ satisfies $\dot{f} : \check{A} \to \check{B}$ is a function.

By the forcing theorem, there is a $p \in G$ such that $p \Vdash \dot{f} : \check{A} \to \check{B}$. Now, define the function F as follows:

$$F(a) = \{b \in B : \exists q \leq p \; q \Vdash \dot{f}(a) = b\}$$

By the definability lemma, the function F belongs to the model M. To prove that $f(a) \in F(a)$. Let $b = f(a)$, then $M[G] \models \dot{f}(\check{a}) = \check{b}$.

Thus, there exists $q \leq p$, $q \in G$ such that $q \Vdash \dot{f}(\check{a}) = \check{b}$ and by the Definition of F we get $b \in F(a)$. Finally, to prove that $(|F(a)| \leq \aleph_0)^M$ let $b \in F(a)$ and choose $q_b \leq p$ so that $q_b \Vdash \dot{f}(\check{a}) = \check{b}$, the the set $\{q_b : b \in F(a)\}$ is an antichain.

Furthermore, the function $b \mapsto q_b$ may be assumed to lie in M. Now, in M $|F(a)| \leq \aleph_0$ because of the chain condition. □

Now we proceed with the main proof.

Proof of the theorem: By the lemma, it is enough to fix a limit ordinal $\omega < \beta < o(M)$ and assume that (β is regular in $)^M$ and prove that this still holds in $M[G]$.

Suppose this is not the case. Then there is $X \subseteq \beta$ with $X \in M[G]$, $\sup(X) = \beta$ and $\alpha := \mathrm{ot}(X) < \beta$. Let $f : \alpha \to X$ be the unique order preserving bijective map.

Then $f : \alpha \to \beta$ so by the lemma before we can find a function $F \in M$, $F : \alpha \to \mathcal{P}(\beta)$. Let $Y = \bigcup_{\gamma<\alpha} F(\gamma)$, this is a subset of β such that $\sup(Y) = \beta$.

In M the set Y is the union of fewer than β countable sets, so $|Y| < \beta$. □

Recall that if we force with $\mathrm{Fn}(\kappa \times \omega, 2)$ we add a sequence $(h_\alpha : \alpha < \kappa)$ such that $h_\alpha \in 2^\omega$. So in the corresponding generic extension $2^{\aleph_0} \geq \kappa$.

The following results aim to compute the exact value of $2^{\aleph}$ in $M[G]$. For this purpose, we define the concept of a *nice name.*

Definition 59.

For $\tau \in V^{\mathbb{P}}$, a nice name for a subset of τ is a name of the form $\bigcup\{\{\sigma\} \times A_\sigma : \sigma \in \mathrm{dom}(\tau)\}$

References

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