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"Some cardinal invariants of the generalized Baire spaces"

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Para mi mami

Abstract

The central theme of the research in this dissertation is the well-known *Cardinal invariants of the continuum*. This thesis consists of two main parts which present the results obtained in joint work with (alphabetically): Jörg Brendle, Andrew Brooke-Taylor, Vera Fischer, Sy-David Friedman and Diego Mejía.

The first part focuses on the generalization of the classical cardinal invariants of the continuum to the generalized Baire spaces κ^κ, when κ is a regular uncountable cardinal. First, we present a generalization of some of the cardinals in Cichoń's diagram to this context and some of the ZFC relationships that are provable between them. Further, we study their values in some generic extensions corresponding to $<\kappa$-support and κ-support iterations of generalized classical forcing notions. We point out the similarities and differences with the classical case and explain the limitations of the classical methods when aiming for such generalizations. Second, we study a specific model where the ultrafilter number at κ is small, 2^κ is large and in which a larger family of cardinal invariants can be decided and proven to be $<2^\kappa$.

The second part focuses exclusively on the countable case: We present a generalization of the method of matrix iterations to find models where various constellations in Cichoń's diagram can be obtained and the value of the almost disjointness number can be decided. The method allows us also to find a generic extension where seven cardinals in Cichoń's diagram can be separated.

Zusammenfassung

Diese Dissertation befasst sich mit den bekannten *Kardinalzahlinvarianten des Kontinuums*: Sie besteht aus zwei Hauptbestandteilen, in der Resulate vorgestellt werden, die in gemeinsamer Arbeit mit (in alphabetischer Reihenfolge) Jörg Brendle, Andrew Brooke-Taylor, Vera Fischer, Sy-David Friedman und Diego Mejía erzielt wurden.

Der erste Teil dieser Arbeit beschäftigt sich mit der Verallgemeinerung der klassischen Kardinalzahlinvarianten des Kontinuums zu den verallgemeinerten Baire-Räumen κ^κ, wobei κ eine überabzählbare reguläre Kardinalzahl ist. Zuerst präsentieren wir eine Verallgemeinerung einiger Kardinalzahlen im Cichoń-Diagramm in diesen Kontext und einige der ZFC-Beziehungen, die zwischen ihnen gelten. Darüber hinaus untersuchen wir ihre Werte in einigen generischen Erweiterungen mittels $<\kappa$-support- und κ-support-Iterationen von verallgemeinerten klassischen Forcings. Wir weisen auf die Ähnlichkeiten und Unterschiede zu dem klassischen Fall hin und gehen auch auf die Einschränkungen der klassischen Methoden im verallgemeinerten Fall ein. Außerdem studieren wir ein bestimmtes Modell, bei dem die Ultrafilterzahl für κ klein ist, während gleichzeitig 2^κ groß ist und in der auch einige andere Kardinalzahlinvarianten diesen Wert annehmen.

Im zweiten Teil konzentrieren wir uns ausschließlich auf den abzählbaren Fall: Wir stellen eine Verallgemeinerung der Methode der Matrix-Iterationen dar um Modelle zu finden, bei denen verschiedene Konstellationen der Kardinalzahlen im Cichoń-Diagramm zusammen mit der *almost disjointness number* erhalten werden können. Die Methode erlaubt uns auch, eine generische Erweiterung zu finden, in der sieben Kardinalzahlen im Cichoń-Diagramm unterschiedliche Werte annehmen.

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Introduction

After Cantor's revolutionary discovery regarding the cardinality of the set of real numbers which states that the size of the set of real numbers is not countable [Can74]; the first cardinal invariant we know appeared, it is precisely the cardinality of the set of reals $\mathfrak{c} = |\mathbb{R}|$, or as many call it, the *cardinality of the continuum*. One main question that arose after Cantor's work in real analysis, for example, was whether or not some properties of the real line that were known to be valid for countable many subsets of the reals could be extended to $\mathfrak{c}$-many such sets.

The continuum problem, i.e. whether the cardinality of the continuum coincides with the first uncountable cardinal ($\mathfrak{c} = \aleph_1$) also played a crucial role in this study. As we now know, this assertion is independent of the axiomatic system ZFC (work by Gödel and Cohen), thus we have the following possibilities: If true, the problem above is bounded to the duality countable-uncountable. In the other scenario, however, one can isolate uncountable cardinals $< \mathfrak{c}$.

Cardinal invariants (or characteristics) of the continuum are cardinals describing mostly the combinatorial or topological structure of the real line. They intend to answer questions like the following: how many meager sets do we need to cover the real line?, how "big" can Lebesgue measure zero sets be? As described in the famous paper of Andreas Blass *Combinatorial cardinal characteristics of the continuum* [Bla10] "... they are simply the smallest cardinals $\leq \mathfrak{c}$ for which various results, true for $\aleph_0$, become false...". They are usually defined in terms of ideals on the reals, or some very closely related structure such as $\mathcal{P}(\omega)/\,\text{fin}$ and typically they assume values between $\aleph_1$, the first uncountable cardinal and $\mathfrak{c}$. Hence, they are uninteresting in models where CH holds. However, as we mentioned above in models of set theory where the continuum hypothesis fails they may assume different values and interact with each other in several ways.

The most common approach when studying these cardinals is to try to answer the following question: which relationships between such cardinals are provable in ZFC and which ones are independent? One particular example of this study that it is central in this work corresponds to the invariants in Cichoń's Diagram (see Figure 0.1). In this diagram, all the possible ZFC relations between some cardinals associated to measure, category and compactness are summarized; today it is known that all the inequalities between two cardinals not contained in the diagram are independent. Moreover, there are models where different arrangements of more than two cardinals (or *constellations*) are proved to be consistent. Furthermore, one of the most engaging open question in the subject nowadays asks whether it is possible to find a generic extension where ten cardinals in the diagram are different.

A natural question that emerges and that motivates the results in chapters 2 and 3 of this dissertation investigates how these invariants can be generalized to the uncountable Baire spaces κ^κ, where κ is an uncountable regular cardinal. Since 1995, with the paper *"Cardinal invariants above the continuum "* from James Cummings and Saharon Shelah [CS95], the study of the generalization of these cardinal notions to the context of uncountable cardinals and their interactions has been developing. By now, there is a wide literature on this topic. Some important references for the purposes of the first two chapters of this dissertation are [BHZ], [CS95] and [Suz98].

The uncountable case happens to be extremely interesting and sometimes really different from the countable one. Namely, some examples that exemplify this phenomenon are the following: first, as mentioned above classic cardinal invariants typically take values in the interval $[\aleph_1, \mathfrak{c}]$. Nevertheless in the uncountable, for example, the straightforward generalization of the classical splitting number $\mathfrak{s}(\kappa)$ can be $\leq \kappa$, and actually large cardinals are necessary to have the expected inequality $\mathfrak{s}(\kappa) \geq \kappa^+$ (see Suzuki [Suz98]). Second, there are new ZFC results that in the countable case do not exist: Raghavan and Shelah [RS15] showed that, for uncountable κ, the inequality $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa)$ holds whereas in the countable case, there are two different forcing extensions in which inequalities $\mathfrak{s} < \mathfrak{b}$ and $\mathfrak{b} > \mathfrak{s}$ hold respectively. As a final example, one can mention Roitman's problem which asks whether from $\mathfrak{d} = \aleph_1$ it is possible to prove that $\mathfrak{a} = \aleph_1$. So far, Shelah gave the best approximation to an answer to this problem: he developed the method of template iteration forcing to give a model in which the inequality $\mathfrak{d} < \mathfrak{a}$ is satisfied, yet in his model the value of $\mathfrak{d}$ is $\aleph_2$; the question that is still open asks if it is possible to find such a model but in addition having $\mathfrak{d} = \aleph_1$. In the uncountable in contrast, Blass, Hyttinen and Zhang [BHZ] proved in ZFC that for uncountable regular κ Roitman's problem can be solved on the positive, namely if $\mathfrak{d}(\kappa) = \kappa^+$, then $\mathfrak{a}(\kappa) = \kappa^+$.

The main results of this thesis were obtained in joint work with Andrew Brooke-Taylor, Jörg Brendle, Vera Fischer, Sy-David Friedman and Diego Mejía and can be found in: [Bre+16; Fis+17; Fis+16]. This thesis consists of two main parts which present results involving cardinal invariants in both the uncountable and the countable cases respectively and it is organized as follows: The preliminaries chapter establishes both the notation and the necessary definitions and background results that are used in the whole document. Chapter 1 presents an attempt of a generalization of Cichoń's Diagram for uncountable cardinals. It provides a review of the basic theory and results, specifically which of the basic inequalities are still ZFC theorems and under which conditions it is possible to obtain them. Particularly, we show that if κ is strongly inaccessible one can obtain a reasonable approximation of the diagram. In addition, we study the values of the cardinals in our generalized diagram in several forcing extensions which are obtained as products, $< \kappa$-support and κ-support iterations of generalizations of classical forcing notions; for instance Cohen forcing, Hechler forcing, localization forcing, Sacks forcing and Miller forcing among others.

Chapter 2 studies the ultrafilter number for uncountable κ, when κ is a supercompact cardinal. It presents a model, which is a modification of a construction of Džamonja and Shelah in [DS03] and allows us to construct a cardinal preserving generic extension obtained from a special iteration of posets in which we prove $\mathfrak{u}(\kappa) = \kappa^*$, where κ^* is a regular cardinal $\kappa^* > \kappa$ and $2^\kappa > \kappa^*$. Besides, our construction allows us also to decide the values of many of the higher analogs of various classical cardinal characteristics of the continuum (including the ones in our generalized version of Cichoń's diagram), by interleaving arbitrary κ-directed closed posets cofinally in the iteration.

The last chapter (Chapter 3) deals exclusively with the cardinal invariants in the countable case and tries to answer the following question: Given a constellation of cardinals in Cichoń's diagram, is it possible to decide in addition the value of the almost disjointness number $\mathfrak{a}$? This chapter presents a generalization of the method of matrix iterations introduced by Blass and Shelah in [BS89] called *coherent systems of finite support iterations* and provides generic extensions where many constellations in Cichoń's diagram can be decided and also $\mathfrak{a} = \mathfrak{b}$. In order to achieve the last part, we extend the preservation machinery developed by Brendle and Fischer in [BF11] regarding preservation of maximal almost disjoint families added by Hechler's poset. The method of coherent systems allows us also to find a generic extension where seven cardinals in the classical Cichoń's diagram can be separated.

At the end of each chapter, we address a list of open questions related to each specific topic which could lead the future research on this topic.

Preliminaries

This chapter aims to give an overview of some important definitions and results that will be used in this document.

0.1. Forcing

The method of forcing was introduced by Paul Cohen in his proof of the independence of the Continuum Hypothesis (CH) and of the Axiom of Choice (AC); specifically, he showed that AC cannot be proved in ZF and CH cannot be proved in (ZFC). Forcing is a general technique for producing a large number of models of ZF and consistency results, and it has become central to the development of set theory. We give a short exposition of the main results involving this technique and its main properties. For technical details and proofs regarding the method we refer the reader to [Kun80] or [Jec03].

Let M be a transitive model of ZFC, from now on *the ground model*. In M, let us consider a nonempty partially ordered set $(\mathbb{P}, <)$. We call $\mathbb{P}$ a *notion of forcing* and the elements of $\mathbb{P}$ *forcing conditions*. We say that p is *stronger* than q if $p \leq q$. If p and q are conditions and there exists r such that both $r \leq p$ and $r \leq q$ we say that p and q are *compatible* and write $p \parallel q$. A set $A \subseteq \mathbb{P}$ is an *antichain* if all its elements are pairwise incompatible. A set $D \subseteq \mathbb{P}$ is *dense* if for every $p \in \mathbb{P}$ there is $q \in D$ stronger than p.

Definition 0.1 (Genericity)**.**

- A nonempty family G of conditions is called a *filter* on $\mathbb{P}$ if given $p \in G$ all conditions $q \leq p$ also belong to G and for every $p, q \in G$ there exists $r \in G$ which satisfies $r \leq p$ and $r \leq q$.
- If M is a model of ZFC containing $\mathbb{P}$, we say that a filter G is $\mathbb{P}$-*generic over M* if $G \cap D \neq \emptyset$ for all D dense subset of $\mathbb{P}$ belonging to M.

Genericity can be described in several different equivalent ways that we will use thoughout this document without distinction depending on the situation. Particularly, instead of demanding that a generic filter G over a model M intersects all dense sets in M, we can instead require that G intersects all the sets of the following types:

- A set $D \subseteq \mathbb{P}$ is *open dense* if it is dense and in addition for all $p \in D$, if $q \leq p$ then $q \in D$.
- A set D is *predense* if every $p \in \mathbb{P}$ is compatible with some $q \in D$.
- A set $D \subseteq \mathbb{P}$ is *dense* (open dense, predense, an antichain) *below a condition* $p \in \mathbb{P}$, if it is dense (open dense, predense, an antichain) in the set $\{q \in \mathbb{P} : q \leq p\}$.

Generic sets do not exist in general, however if the ground model M is countable they do:

Lemma 0.2 (Existence of generics over countable models)**.** Let $\mathbb{P}$ be a forcing notion and $\mathcal{D}$ be a countable collection of dense subsets of $\mathbb{P}$, then there exists a $\mathcal{D}$-generic filter on $\mathbb{P}$. In fact, for every $p \in \mathbb{P}$ there exists a $\mathcal{D}$ generic filter containing p.

The first fundamental theorem of the forcing method states the following:

Theorem 0.3. *[Jec03] Let M be a model of* ZFC *containing $\mathbb{P}$, and let $G \subseteq \mathbb{P}$ be a generic filter over M. Then there is a model $M[G]$ of* ZFC *which includes $M \cup \{G\}$, has the same ordinals as M, and which is minimal in the sense that if W is any model of* ZFC *including $M \cup \{G\}$, then $M[G] \subseteq W$.*

The model $M[G]$ is called a *generic extension* of V. The sets in $M[G]$ are definable from G and finitely many elements of M. Namely, $M^{\mathbb{P}}$ is the class of $\mathbb{P}$-*names* over M, $M^{\mathbb{P}} = \bigcup_{\alpha \in M \cap \mathrm{OR}} M^{\mathbb{P}}_{\alpha}$ where $M^{\mathbb{P}}_{\alpha} = \{\tau \in M : \tau$ is a binary relation and $\forall (\sigma, p) \in \tau (p \in \mathbb{P} \wedge \exists \beta < \alpha\, (\sigma \in M^{\mathbb{P}}_{\beta}))\}$ and $M[G]$ consist of the *interpretation* of the class of names according to the generic filter G:

Definition 0.4. Let M be a transitive model of ZFC, and let $\mathbb{P}$ be a forcing notion and G be a $\mathbb{P}$-generic filter over M. For $\tau \in M^{\mathbb{P}}$ we define $\tau^G = \{\sigma^G : \exists p \in G (\sigma, p) \in \tau\}$ and $M[G] = \{\tau^G : \tau \in M^{\mathbb{P}}\}$.

An important feature of the generic extension is that it can be described within the ground model. Also, associated with the notion of forcing $\mathbb{P}$ there is a *forcing language* and a *forcing relation* $\Vdash$, both of them are also defined in the ground model M. The forcing relation $\Vdash$ is a relation between the forcing conditions and sentences of the forcing language, we write $p \Vdash \sigma$ and read *p forces* σ. This relation is a generalization of the notion of satisfaction. For instance, if $p \Vdash \sigma$ and if σ' is a logical consequence of σ, then $p \Vdash \sigma'$.

The second fundamental theorem on generic models establishes the relation between the forcing relation and truth in the model $M[G]$:

Theorem 0.5. *[Jec03] Let $\mathbb{P}$ be a forcing notion in the ground model M. If σ is a sentence of the forcing language, then for every generic $G \subseteq \mathbb{P}$ over M.*

$$M[G] \models \sigma \text{ if and only if } (\exists p \in G)(p \Vdash \sigma)$$

Where "$M \models \sigma$ "means that one interprets the constants of the forcing language according to G.

The forcing relation has the following important basic properties:

Theorem 0.6. *Let $\mathbb{P}$ be a forcing notion in the ground model M, and let $M^{\mathbb{P}}$ be the sets of names. Then:*

1. - *If $p \Vdash \varphi$ and $q \leq p$, then $q \Vdash \varphi$.*
   - *No p forces both φ and $\neg\varphi$.*
   - *For every p there exists $q \leq p$ deciding φ, i.e. $q \Vdash \varphi$ or $q \Vdash \neg\varphi$.*
2. - *$p \Vdash \neg\varphi$ if and only if no $q \leq p$ forces φ.*
   - *$p \Vdash \varphi \wedge \psi$ if and only if $p \Vdash \varphi$ and $p \Vdash \psi$. $p \Vdash \forall x\varphi$ if and only if $p \Vdash \varphi(\dot{a})$ for every $\dot{a}$ in $M^{\mathbb{P}}$.*
   - *$p \Vdash \varphi \vee \psi$ if and only if for all $q \leq p$ there exists $r \leq q$ such that $r \Vdash \varphi$ or $r \Vdash \psi$. $p \Vdash \exists x\varphi$ if and only if for all $q \leq p$ there exists $r \leq q$ and $\dot{a} \in M^{\mathbb{P}}$ such that $r \Vdash \varphi(\dot{a})$.*
3. *If $p \Vdash \exists x\varphi$ then for some $\dot{a} \in M^{\mathbb{P}}$, $p \Vdash \varphi(\dot{a})$.*

0.1.1. Products and iterated forcing

Usually, in the applications of the forcing method, one wants to add more than one generic object simultaneously. That motivates the following classic methods to do so, we present a small overview of them.

Products

Let $\mathbb{P}$ and $\mathbb{Q}$ be two forcing notions, the *product* $\mathbb{P} \times \mathbb{Q}$ is the coordinatewise partially ordered set product of $\mathbb{P}$ and $\mathbb{Q}$, $(p_2, q_2) \leq (p_1, q_1)$ if and only if $p_2 \leq p_1$ and $q_2 \leq q_1$. Then if G is a $\mathbb{P} \times \mathbb{Q}$-generic filter then $G_1 = \pi_1(G)$ and $G_2 = \pi_2(G)$ are $\mathbb{P}$ and $\mathbb{Q}$ generic respectively (here π_1 and π_2 are the projections on the first and second coordinate respectively). The following lemma describes genericity on products:

Lemma 0.7. $G \subseteq \mathbb{P} \times \mathbb{Q}$ is generic over M if and only if $G = G_1 \times G_2$ where G_1 is $\mathbb{P}$-generic over M and G_2 is $\mathbb{Q}$-generic over $M[G_1]$. Moreover $M[G] = M[G_1][G_2]$.

Definition 0.8. Let $\{\mathbb{P}_i : i \in I\}$ be a collection of forcing notions, each having greatest element $\mathbb{1}$. The product $\mathbb{P} = \prod_{i \in I} \mathbb{P}_i$ consists of all functions $p : I \to \dot{\bigcup}_{i \in I} \mathbb{P}_i$ with $p(i) \in \mathbb{P}_i$ ordered by $p \leq q$ if and only if $p(i) \leq q(i)$ for all $i \in I$.

Given $p \in \mathbb{P}$, the *support* of the condition p is defined by $\operatorname{supp}(p) = \{i \in I : p(i) \neq \mathbb{1}\}$. We say that $\mathbb{P}$ is a *κ-product* or *product with $<!\kappa$-support* if $|\operatorname{supp}(p)| < \kappa$ for all $p \in \mathbb{P}$.

Iterations

Let $\mathbb{P}$ be a forcing notion and let $\dot{\mathbb{Q}} \in M^{\mathbb{P}}$ be a $\mathbb{P}$-name for a partial order in $M^{\mathbb{P}}$. We define the two-step iteration forcing by $\mathbb{P} * \dot{\mathbb{Q}} = \{(p, \dot{q}) : p \in \mathbb{P}$ and $\Vdash_{\mathbb{P}} \dot{q} \in \dot{Q}\}$ ordered as follows $(p_1, \dot{q}_1) \leq (p_2, \dot{q}_2)$ if and only if $p_1 \leq p_2$ and $p_1 \Vdash \dot{q}_1 \leq \dot{q}_2$.

Definition 0.9. Let $\alpha \geq 1$. A forcing notion $\mathbb{P}_\alpha$ is an *iteration* of length α if it is the set of α-sequences with the following properties:

1. If $\alpha = 1$ then for some forcing notion $\mathbb{Q}_0$, $\mathbb{P}_1$ is the set of all sequences $(p(0))$ where $p(0) \in \mathbb{Q}_0$, ordered by $(p(0)) \leq_1 (q(0))$ if and only if $p(0) \leq_{\mathbb{Q}_0} q(0)$.
2. If $\alpha = \beta + 1$ then $\mathbb{P}_\beta = \mathbb{P}_\alpha \upharpoonright \beta = \{p \upharpoonright \beta : p \in \mathbb{P}_\alpha\}$ is an iteration of length β, and there is some forcing notion $\dot{\mathbb{Q}}_\beta \in M^{\mathbb{P}_\beta}$ such that $p \in \mathbb{P}_\alpha$ if $p \upharpoonright \beta \in \mathbb{P}_\beta$ and $\Vdash_\beta p(\beta) \in \dot{\mathbb{Q}}_\beta$ and $p \leq_\alpha q$ if and only if $p \upharpoonright \beta \leq_\beta q \upharpoonright \beta$ and $p \upharpoonright \beta \Vdash_\beta p(\beta) \leq q(\beta)$.
3. If α is a limit ordinal then $\mathbb{P}_\beta = \mathbb{P}_\alpha \upharpoonright \beta = \{p \upharpoonright \beta : p \in \mathbb{P}_\alpha\}$ is an iteration of length β for every $\beta < \alpha$ and $(1, 1, \ldots) \in \mathbb{P}_\alpha$; if $p \in \mathbb{P}_\alpha$, $\beta < \alpha$ and $q \in \mathbb{P}_\beta$ is such that $q \leq_\beta p \upharpoonright \beta$, then $r \in \mathbb{P}_\alpha$ where for all $\xi < \alpha$, $r(\xi) = q(\xi)$ if $\xi < \beta$ and $r(\xi) = p(\xi)$ is $\beta \leq \xi < \alpha$ and $p \leq_\alpha q$ if and only if $p \upharpoonright \beta \leq_\beta q \upharpoonright \beta$ for all $\beta < \alpha$.

A general iteration depends not only on $\dot{\mathbb{Q}}_\beta$ but also on the limit steps of the iteration. Let $\mathbb{P}_\alpha$ be an iteration of length α limit, we say that $\mathbb{P}_\alpha$ is a direct limit if given any α-sequence p, $p \in \mathbb{P}_\alpha$ if and only if $\exists \beta < \alpha$ such that $p \upharpoonright \beta \in \mathbb{P}_\beta$ and $\forall \xi \geq \beta\ p(\xi) = 1$.

We also say that $\mathbb{P}_\alpha$ is an inverse limit if given any α-sequence p, $p \in \mathbb{P}_\alpha$ if and only if for all $\beta < \alpha$, $p \upharpoonright \beta \in \mathbb{P}_\beta$.

Forcing iterations combine both direct and inverse limits, the most common are the *finite support iteration* where direct limits are taken at limit stages and the *countable support iteration* in which inverse limits are taken at all limit steps of countable cofinality and direct limits elsewhere.

0.1.2. Some specific types of forcing notions

Throughout this document, forcing extensions that preserve cardinals will be crucial, i.e. we are interested in forcing notion $\mathbb{P}$, such that whenever κ is a cardinal in the ground model V, then $\kappa^{V^{\mathbb{P}}} = \kappa^V$. That is why we need to use several properties that are helpful in order to ensure this property for our future forcing constructions.

Definition 0.10. Let $\mathbb{P}$ be a forcing notion, we say that $\mathbb{P}$ is:

1. *κ-cc* if any maximal antichain in $\mathbb{P}$ has size $<\kappa$, when $\kappa = \omega$ we say that $\mathbb{P}$ is ccc.
2. *κ- centered* if $\mathbb{P} = \bigcup_{\alpha<\kappa} \mathbb{P}_\alpha$ where for every $\alpha < \kappa$, if $p, q \in \mathbb{P}_\alpha$ there exists $r \in \mathbb{P}_\alpha$ such that $r \leq p$ and $r \leq q$.
3. *κ-closed* if for every decreasing sequence $(p_\alpha : \alpha < \gamma)$, $\gamma < \kappa$ there exists $p \in \mathbb{P}$, such that $p \leq p_\alpha$ for all $\alpha < \gamma$.
4. *κ-distributive* if the intersection of less than κ many open dense subsets of $\mathbb{P}$ is open dense.
5. *κ-directed closed* if whenever $D \subseteq \mathbb{P}$ is such that $|D| < \kappa$ and for every $p_1, p_2 \in D$ there exists $r \in D$ with $r \leq p_1$ and $r \leq p_2$, then there is a $q \in \mathbb{P}$ such that $q \leq p$, for all $p \in D$.

6. *proper* if for all large enough regular cardinals χ, all countable models $N \prec H(\chi)$ with $\mathbb{P} \in N$, and all conditions $p \in \mathbb{P} \cap N$, there is $q \leq p$ which is $(N, \mathbb{P})$-generic. This means that for all dense sets $D \subseteq \mathbb{P}$ in M, $D \cap N$ is predense below q.
7. of *precaliber* $\aleph_1$ if for all $X \subseteq \mathbb{P}$ with $|X| = \aleph_1$ there exists $X' \in [X]^{\aleph_1}$ such that all its finite sets are compatible, i.e. given $F \subseteq X$ finite, there exists a condition $p \in \mathbb{P}$ extending all its elements.

For preservation of cardinal purposes, the results below will be essential:

Theorem 0.11 (Theorems 15.3 and 15.6 in [Jec03])**.**

1. *If κ is a regular cardinal and $\mathbb{P}$ satisfies the κ-cc, then κ-remains a regular cardinal in the generic extension by $\mathbb{P}$. Consequently, all regular cardinals $\kappa \geq |\mathbb{P}|^+$ are preserved.*
2. *Let κ be an infinite cardinal and assume that $\mathbb{P}$ is κ^+-closed. Then if $f \in V[G]$ is a function from κ to V, then $f \in V$. In particular, κ has no new subsets in $V[G]$.*

Theorem 0.12 (Theorem 4.3, Corollary 4.5 in [Bre09])**.** *If $\mathbb{P}$ is proper, then $\mathbb{P}$ preserves $\aleph_1$.*

Theorem 0.13 (15.15 in [Jec03])**.**

1. *If each $\mathbb{P}_i$ has size λ (infinite), then the κ-product of the $\mathbb{P}_i$ satisfies the λ^+-chain condition.*
2. *If κ is regular, $\lambda \geq \kappa$ and $\lambda^{<\kappa} = \lambda$, $|\mathbb{P}_i| \leq \lambda$ for all $i \in I$, then the κ-product of the $\mathbb{P}_i$ satisfies the λ^+-chain condition.*
3. *If λ is inaccessible, $\kappa < \lambda$ is regular and $|\mathbb{P}_i| < \lambda$ for each i, then the κ-product of the $\mathbb{P}_i$ satisfies the λ-chain condition.*

Theorem 0.14 (Shelah, see Theorem 31.15 in [Jec03])**.** *If $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta)$ is a countable support iteration such that $\Vdash_\alpha$ "$\dot{\mathbb{Q}}_\alpha$ is proper", then $\mathbb{P}_\delta$ is proper.*

Classical forcing notions adding reals

We present here a list of some forcing notions that add reals and which are used to find models where some constellations of cardinal invariants can be realized. We list particularly the forcings we will use in the subsequent chapters.

1. Countable chain forcings that are usually iterated with finite support.[1]
   - ***Cohen forcing*** $\mathbb{C}$: Is simply the poset whose conditions are functions in $2^{<\omega}$ ordered by reverse inclusion. It can be also expressed as the quotient $\mathcal{B}(2^\omega)/\mathcal{M}$, where $\mathcal{M}$ is the σ-ideal of meager sets of 2^κ ordered by $[X] \leq [Y]$ if and only if $X \setminus Y$ is meager.
   - ***Hechler forcing*** $\mathbb{D}$: Conditions have the form (s, f), where $s \in \omega^{<\omega}$ and $f \in \omega^\omega$ with the order given by, $(s, f) \leq (t, g) \leftrightarrow s \supseteq t$, f dominates g everywhere (i.e. $\forall n\ (g(n) < f(n))$) and $\forall m \in \mathrm{dom}(s) \setminus \mathrm{dom}(t)$, $s(m) \geq g(m)$.

[1] Random forcing, for instance, can be also iterated with countable support

- ***Random forcing*** $\mathbb{B}$: Corresponds to the quotient algebra $\mathcal{B}(2^\omega)/\mathcal{N}$, where $\mathcal{N}$ is the σ-ideal of null sets with respect to the standard product measure on 2^ω, ordered by inclusion modulo the ideal. More generally, if Ω is a non-empty countable set, $\mathbb{B}_\Omega$ is the complete Boolean algebra $2^{\Omega\times\omega}/\mathcal{N}(2^{\Omega\times\omega})$ where the σ-ideal $\mathcal{N}(2^{\Omega\times\omega})$ is defined analogously. Clearly $\mathbb{B}_\Omega \simeq \mathbb{B} := \mathbb{B}_\omega$ and we have that for any set Γ, $\mathbb{B}_\Gamma := \mathrm{limdir}\{\mathbb{B}_\Omega : \Omega \subseteq \Gamma \text{ and } \Omega \text{ is countable}\}$. We denote by $\mathfrak{R}$ the class of all random algebras, that is, $\mathfrak{R} := \{\mathbb{B}_\Gamma : \Gamma \neq \emptyset\}$.
- ***Eventually different forcing*** $\mathbb{E}$: Conditions have the form (s, F) where $s \in \omega^{<\omega}$ and $F \in [\omega^\omega]^{<\omega}$ with the order given by: $(s, F) \leq (t, G)$ if and only if $s \supseteq t$, $F \supseteq G$ and $\forall g \in G\ \forall m \in \mathrm{dom}(t) \setminus \mathrm{dom}(s)$, $s(m) \neq g(m)$.

  Another forcing adding an eventually different real $\mathbb{E}^*$: Conditions of the form (s, φ) where $s \in \omega^{<\omega}$ and $\varphi : \omega \to [\omega]^{<\omega}$ such that $\exists n < \omega \forall i < \omega(|\varphi(i)| \leq n)$. The minimal such n is denoted by $\mathrm{width}(\varphi)$. The order is defined as $(t, \psi) \leq (s, \varphi)$ if and only if $s \subseteq t$, $\forall i < \omega(\varphi(i) \subseteq \psi(i))$ and $\forall i \in |t| \setminus |s|(t(i) \notin \varphi(i))$.
- ***Localization forcing*** $\mathbb{LOC}$: Conditions are functions $\varphi \in ([\omega]^{<\omega})^\omega$ such that for all $n \in \omega$, $|\varphi(n)| \leq n$ and there is $k \in \omega$ such that for all but finitely many n, $|\varphi(n)| \leq k$. The order is defined as $\varphi' \leq \varphi$ if and only if $\varphi(n) \subseteq \varphi'(n)$ for all $n < \omega$.
- ***Mathias-Příkrý Forcing*** $\mathbb{M}_\mathcal{U}$: Let $\mathcal{U}$ be an ultrafilter on ω, conditions in this forcing have the form (s, A) where $s \in [\omega]^{<\omega}$ and $A \in \mathcal{U}$. The ordering is given by: $(t, B) \leq (s, A)$ if and only if $t \sqsupseteq s$, $B \subseteq A$ and $t \setminus s \subseteq A$.

2. Tree forcings that have good fusions and therefore can be iterated with countable support.

   Recall that a *tree* is a partially ordered set $(T, <)$ with the property that for each $t \in T$, the set $\{s : s < t\}$ is well-ordered by $<$. The *stem* of T is unique splitting node of T that is related (via $<$) with all elements in T.

   - ***Sacks forcing*** $\mathbb{S}$: Conditions are perfect nonempty subtrees $T \subseteq 2^{<\omega}$, meaning that for every $t \in T$, there exists $s \supseteq t$, $s \in T$ such that both $s^\frown 0$ and $s^\frown 1$ belong to T. The order is inclusion, i.e. $T \leq S$ if $T \subseteq S$.
   - ***Miller forcing*** $\mathbb{M}$: Conditions are nonempty subtrees $T \subseteq \omega^{<\omega}$, such that for every $t \in T$ above $\mathrm{stem}(T)$, there exists $s \supseteq t$, $s \in T$ such that for infinitely many $n \in \omega$ $s^\frown n \in T$. The order is also inclusion.
   - ***Laver forcing*** $\mathbb{L}$: Conditions are nonempty subtrees $T \subseteq \omega^{<\omega}$, such that for every $t \in T$ above $\mathrm{stem}(T)$ there are infinitely many $n \in \omega$ with $t^\frown n \in T$. The order is again inclusion.

0.2. Cardinal invariants of the continuum

As mentioned in the introduction, cardinal invariants of the continuum are cardinals describing mostly the combinatorial or topological structure of the real line. We define some of them, that will be used in the following chapters.

Definition 0.15. Some special subsets of the reals:

- If f, g are functions from ω to ω, we say that $f \leq^* g$, if there exists an $n \in \omega$ such that for all $m > n$, $f(m) \leq g(m)$. In this case, we say that *g eventually dominates f*.
- Let $\mathfrak{F} \subseteq \omega^\omega$, we say that $\mathfrak{F}$ is *dominating*, if for all $g \in \omega^\omega$, there exists an $f \in \mathfrak{F}$ such that $g \leq^* f$. $\mathfrak{F} \subseteq \kappa^\kappa$ is *unbounded*, if for all $g \in \omega^\omega$ there exists an $f \in \mathfrak{F}$ such that $f \nleq^* g$.
- For $A, B \in \mathcal{P}(\omega)$, say $A \subseteq^* B$ (A is *almost contained* in B) if $A \setminus B$ is finite. We also say that A *splits* B if both $A \cap B$ and $B \setminus A$ are infinite. A family $\mathcal{A}$ is called a *splitting family* if every infinite subset of ω is split by a member of $\mathcal{A}$. Finally $\mathcal{A}$ is *unsplit* if no single set splits all members of $\mathcal{A}$.
- Two sets A and $B \in \mathcal{P}(\omega)$ are called *almost disjoint* if $A \cap B$ is finite. We say that a family of sets $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is *almost disjoint* if all its elements are pairwise almost disjoint. Finally, we say that a family $\mathcal{A} \subseteq [\omega]^\omega$ is *maximal almost disjoint (mad)* if it is not properly included in another such family.
- A family $\mathcal{I} = \{I_\delta : \delta < \mu\}$ of subsets of ω is called *independent* if for all disjoint finite sets $I_0, I_1 \subseteq \mu$, $\bigcap_{\delta \in I_0} I_\delta \cap \bigcap_{\delta \in I_1} (I_\delta)^c$ is infinite.
- We say that a family $\mathcal{F}$ of subsets of ω has the *finite intersection property* (FIP) if any finite subfamily $\mathcal{F}' \subseteq \mathcal{F}$ has infinite intersection, we also say that $A \subseteq \omega$ is a *pseudointersection* of $\mathcal{F}$, $A \subseteq^* F$ for all $F \in \mathcal{F}$. A *tower* $\mathcal{T}$ is a well-ordered family of subsets of ω with the FIP that has no infinite pseudointersection.

Definition 0.16. Some cardinal invariants:

- The *unbounding number*,
  $\mathfrak{b} = \min\{|\mathfrak{F}|$: $\mathfrak{F}$ is an unbounded family of functions from ω to $\omega\}$.
- The *dominating number*,
  $\mathfrak{d} = \min\{|\mathfrak{F}|$: $\mathfrak{F}$ is a dominating family of functions from ω to $\omega\}$.
- The *splitting number*,
  $\mathfrak{s} = \min\{|\mathcal{A}|$: $\mathcal{A}$ is a splitting family of subsets of $\omega\}$.
- The *reaping number*,
  $\mathfrak{r} = \min\{|\mathcal{A}|$: $\mathcal{A}$ is an unsplit family of subsets of $\omega\}$.
- The *almost disjointness number*,
  $\mathfrak{a} = \min\{|\mathcal{A}|$: $\mathcal{A}$ is a mad family of subsets of $\omega\}$.
- The *independence number*,
  $\mathfrak{i} = \min\{|\mathcal{I}|$: $\mathcal{I}$ is an independent family of subsets of $\omega\}$.
- The *pseudointersection number*,
  $\mathfrak{p} = \min\{|\mathcal{F}|$: $\mathcal{F}$ is a family of subsets of ω with the FIP and no infinite pseudointersection $\}$.
- The *tower number*,
  $\mathfrak{t} = \min\{|\mathcal{T}|$: $\mathcal{T}$ is a tower of subsets of $\omega\}$.

- The *distributivity number*,
  $\mathfrak{h} = \min\{\lambda : \mathcal{P}(\omega)/\,\text{fin is not } \lambda\text{-distributive}\}$.

Definition 0.17 (Cardinal Invariants Associated to an Ideal)**.** Let $\mathcal{I}$ be a σ-ideal on a set X, we define:

- The *additivity number*:

$$\operatorname{add}(\mathcal{I}) = \min\{|\mathcal{J}| \colon \mathcal{J} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{J} \notin \mathcal{I}\}.$$

- The *covering number*:

$$\operatorname{cov}(\mathcal{I}) = \min\{|\mathcal{J}| \colon \mathcal{J} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{J} = X\}.$$

- The *cofinality number*:

$$\operatorname{cof}(\mathcal{I}) = \min\{|\mathcal{J}| \colon \mathcal{J} \subseteq \mathcal{I} \text{ and for all } M \in \mathcal{I} \text{ there is a } J \in \mathcal{J} \\ \text{with } M \subseteq J\}.$$

- The *uniformity number*:

$$\operatorname{non}(\mathcal{I}) = \min\{|Y| \colon Y \subset X \text{ and } Y \notin \mathcal{I}\}.$$

Now, endow ω^ω with the product topology which is generated by the basic clopen sets $[s] = \{f \in \omega^\omega : s \subseteq f\}$ where $s \in \omega^{<\omega}$ and let $\mathcal{M}$ be the σ-ideal of meager sets with respect to this topology. Also endow 2 with the measure that gives both $\{0\}$ and $\{1\}$ weight $1/2$ and ω with the measure that gives $\{n\}$ weight $1/2^{n+1}$, and finally consider 2^ω and ω^ω as measure spaces with the respective product measures. Then $\mathcal{N}$ is the ideal of null subsets of 2^ω or ω^ω with respect to the corresponding measure.

Provable ZFC inequalities between the cardinals associated with the meager and null ideals as well as the unbounding and dominating numbers can be summarized in the well-known Cichoń's Diagram. Here, an arrow ($\to$) means ($\leq$).

$$\begin{tikzcd}
& \operatorname{cov}(\mathcal{N}) \arrow[r] & \operatorname{non}(\mathcal{M}) \arrow[r] & \operatorname{cof}(\mathcal{M}) \arrow[r] & \operatorname{cof}(\mathcal{N}) \arrow[r] & \mathfrak{c} \\
& & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & & \\
\aleph_1 \arrow[r] & \operatorname{add}(\mathcal{N}) \arrow[uu] \arrow[r] & \operatorname{add}(\mathcal{M}) \arrow[u] \arrow[r] & \operatorname{cov}(\mathcal{M}) \arrow[u] \arrow[r] & \operatorname{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}$$

Figure 0.1.: Cichoń's diagram

In addition we have the following relations:

Proposition 0.18 (Miller and Truss, See 2.2.9 and 2.2.11 in [BJ95]).

- $\mathrm{cof}(\mathcal{M}) = \max\{\mathrm{non}(\mathcal{M}), \mathfrak{d}\}$.
- $\mathrm{add}(\mathcal{M}) = \min\{\mathrm{cov}(\mathcal{M}), \mathfrak{b}\}$.

Moreover, if we consider the models obtained by iterating the posets defined in 0.1.2 with finite or countable support accordingly, we can calculate the values of the cardinal invariants defined above. The following table summarizes some of these results (see also [Bla10]):

Effect of some classical iterations on some cardinal invariants						
Cardinal	Cohen	Random	Hechler	Sacks	Miller	Laver
$\mathfrak{a}$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$
$\mathfrak{b}$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$
$\mathfrak{d}$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$
$\mathfrak{h}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$
$\mathfrak{i}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$
$\mathfrak{p}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$
$\mathfrak{r}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$
$\mathfrak{s}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$
$\mathrm{add}(\mathcal{N})$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$
$\mathrm{cov}(\mathcal{N})$	$\aleph_1$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$
$\mathrm{non}(\mathcal{N})$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\aleph_1$
$\mathrm{cof}(\mathcal{N})$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$
$\mathrm{add}(\mathcal{M})$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\aleph_1$
$\mathrm{cov}(\mathcal{M})$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\aleph_1$
$\mathrm{non}(\mathcal{M})$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$
$\mathrm{cof}(\mathcal{M})$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$

0.3. Preservation properties for finite support iterations

We review some preservation results for finite iterations developed by Judah and Shelah [JS90] and Brendle [Bre91] that will be used mostly in Chapter 3. A similar presentation of them also appears in [GMS16, Sect. 3].

Definition 0.19. A triple $\mathcal{R} := (X, Y, \sqsubset)$ is a *Polish relational system* if the following conditions are satisfied:

1. X is an uncountable Polish space.
2. Y is a non-empty analytic subspace of some Polish space.
3. $\sqsubset = \bigcup_{n<\omega} \sqsubset_n$ for some increasing sequence $(\sqsubset_n : n < \omega)$ of closed subsets of $X \times Y$ such that $(\sqsubset_n)^y = \{x \in X : x \sqsubset_n y\}$ is nowhere dense for all $y \in Y$.

For $x \in X$ and $y \in Y$, $x \sqsubset y$ is often read y *$\sqsubset$-dominates x*.

Definition 0.20. Let $\mathcal{R} := (X, Y, \sqsubset)$ be a Polish relational system, we say that:

- A family $\mathcal{F} \subseteq X$ is $\mathcal{R}$-*unbounded* if there is no real in Y that $\sqsubset$-dominates every member of $\mathcal{F}$.
- $x \in X$ is $\mathcal{R}$-*unbounded over a model* M if $x \not\sqsubset y$ for all $y \in Y \cap M$.
- Given a cardinal λ, $\mathcal{F} \subseteq X$ is λ-$\mathcal{R}$-*unbounded* if for any $Z \subseteq Y$ of size $<\lambda$, there is an $x \in \mathcal{F}$ that is $\mathcal{R}$-unbounded over Z.
- $\mathcal{D} \subseteq Y$ is a $\mathcal{R}$-*dominating* family if every member of X is $\sqsubset$-dominated by some member of $\mathcal{D}$.
- $\mathfrak{b}(\mathcal{R}) := \min\{|\mathcal{F}| \colon \mathcal{F}$ is $\mathcal{R}$-unbounded$\}$.
- $\mathfrak{d}(\mathcal{R}) := \min\{|\mathcal{D}| \colon \mathcal{D}$ is $\mathcal{R}$-dominating$\}$.

From now on, fix a Polish relational system $\mathcal{R} = (X, Y, \sqsubset)$ and an uncountable regular cardinal θ.

Remark 0.21. It is possible, without loss of generality to assume $Y = \omega^\omega$. There exists a continuous, onto $f : \omega^\omega \to Y$, so that the Polish relational system $\mathcal{R}' := \langle X, \omega^\omega, \sqsubset' \rangle$, where $x \sqsubset'_n z$ if and only if $x \sqsubset_n f(z)$ behaves like $\mathcal{R}$. Furthermore, the notions of λ-$\mathcal{R}$-unbounded and λ-$\mathcal{R}'$-unbounded are equivalent.

Definition 0.22 (Judah and Shelah [JS90])**.** A forcing notion $\mathbb{P}$ is θ-$\sqsubset$-*good* if for any $\mathbb{P}$-name $\dot{h}$ for a real in Y, there is a non-empty $H \subseteq Y$ of size $<\theta$ such that $\Vdash x \not\sqsubset \dot{h}$ for any $x \in X$ that is $\mathcal{R}$-unbounded over H.

If $\mathbb{P}$ is $\aleph_1$-$\mathcal{R}$-good we say that it is $\mathcal{R}$-*good*.

Note that given two different Polish relational systems $\mathcal{R}$ and $\mathcal{R}'$ the notions of θ-$\mathcal{R}$-good and θ-$\mathcal{R}'$-good are equivalent. Also, if $\theta \leq \theta'$ then θ-$\mathcal{R}$-good implies θ'-$\mathcal{R}$-good. Finally, any poset completely embedded into a θ-$\mathcal{R}$-good poset is also θ-$\mathcal{R}$-good.

Forcing notions with this property are quite useful when one wants to preserve $\mathcal{R}$-unbounded families after forcing. Specifically, it holds that any θ-$\mathcal{R}$-good forcing preserves every θ-$\mathcal{R}$-unbounded family from the ground model.

Furthermore, this property is iterable with finite support. This means that finite support iterations of θ-cc, θ-$\mathcal{R}$-good posets turn out to be θ-$\mathcal{R}$-good as well.

Since we preserve $\mathcal{R}$-unbounded families of the ground model, when forcing with this family of posets we can then decide the value of $\mathfrak{b}(\mathcal{R})$ to be small and the value of $\mathfrak{d}(\mathcal{R})$ to be large: if $\mathcal{F}$ is a θ-$\mathcal{R}$-unbounded family, then $\mathfrak{b}(\mathcal{R}) \leq |\mathcal{F}|$ and $\theta \leq \mathfrak{d}(\mathcal{R})$. Now we mention some examples of forcings with this property:

Lemma 0.23 ([Mej13, Lemma 4])**.** Any poset of size $<\theta$ is θ-$\mathcal{R}$-good. In particular, Cohen forcing is $\mathcal{R}$-good.

Proof. Let $\mathbb{P} = \{p_\alpha : \alpha < \lambda\}$ where $\lambda < \theta$ and let $\dot{h}$ be a $\mathbb{P}$-name for a real in ω^ω, then for every $\alpha < \lambda$ there are decreasing sequences $D_\alpha = \{q^\alpha_n : n \in \omega\}$ below the corresponding p_α and functions $h_\alpha \in \omega^\omega$ such that for all $n \in \omega$, $q^\alpha_n \Vdash \dot{h} \restriction n = h_\alpha \restriction n$.

Then, it suffices to prove that for all $f \in \omega^\omega$ such that $f \not\sqsubset h_\alpha$ we have $f \not\sqsubset \dot{h}$. To this end, fix $p \in \mathbb{P}$ and $m \in \omega$ and take $\alpha < \lambda$ with $p = p_\alpha$. Since $f \not\sqsubset h_\alpha$ and $(\sqsubset_m)_f = \{g \in \omega^\omega : f \sqsubset_m g\}$ is closed there is $n \in \omega$ such that $[h_\alpha \restriction n] \cap (\sqsubset_m)_f = \emptyset$ and so $q_n^\alpha \Vdash [\dot{h} \restriction n] \cap (\sqsubset_m)_f = \emptyset$ which implies $q_n^\alpha \Vdash f \sqsubset_m \dot{h}$. □

Example 0.24. The following are important examples that we will use in the next chapters.

1. *Combinatorial characterizations for* $\mathrm{non}(\mathcal{M})$ *and* $\mathrm{cov}(\mathcal{M})$: Consider the Polish relational system $\mathcal{E} := (\omega^\omega, \omega^\omega, \neq^*)$ where $x \neq^* y$ if and only if x and y are eventually different, that is, $x(i) \neq y(i)$ for all but finitely many $i < \omega$. A well-known result of Bartoszyński-Judah shows that $\mathfrak{b}(\mathcal{E}) = \mathrm{non}(\mathcal{M})$ and $\mathfrak{d}(\mathcal{E}) = \mathrm{cov}(\mathcal{M})$ (see [BJ95, Theorem 2.4.1 and 2.4.7]).
2. *Preserving unbounded families:* Let $\mathcal{D}$ be the Polish relational system $\mathcal{D} = (\omega^\omega, \omega^\omega, \leq^*)$. Clearly, $\mathfrak{b}(\mathcal{D}) = \mathfrak{b}$ and $\mathfrak{d}(\mathcal{D}) = \mathfrak{d}$.

   Miller [Mil81] proved that $\mathbb{E}$ is $\mathcal{D}$-good. Besides, ω^ω-bounding posets are $\mathcal{D}$-good, like the random algebras.
3. *Preserving null-covering families:* Let $b : \omega \to \omega \setminus \{0\}$ such that $\sum_{i<\omega} \frac{1}{b(i)} < +\infty$ and let $\mathcal{E}_b := (\mathbb{R}_b, \mathbb{R}_b, \neq^*)$ be the Polish relational system where $\mathbb{R}_b := \prod_{i<\omega} b(i)$. Given $x \in \mathbb{R}_b$, the set $\{y \in \mathbb{R}_b : \neg(x \neq^* y)\}$ has measure zero with respect to the standard Lebesgue measure on $\mathbb{R}_b$, thus the inequalities $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{b}(\mathcal{E}_b)$ and $\mathfrak{d}(\mathcal{E}_b) \leq \mathrm{non}(\mathcal{N})$ hold.

   Moreover, results by Brendle (see [Bre91, Lemma 1*]) show that any ν-centered poset is θ-$\mathcal{E}_b$-good for any $\nu < \theta$ infinite. In particular, σ-centered posets are $\mathcal{E}_b$-good.
4. *Preserving "union of null sets is not null":* For each $k < \omega$ let $\mathrm{id}^k : \omega \to \omega$ such that $\mathrm{id}^k(i) = i^k$ for all $i < \omega$ and put $\mathcal{H} := \{\mathrm{id}^{k+1} : k < \omega\}$. Let $\mathrm{LOC} := (\omega^\omega, \mathcal{S}(\omega, \mathcal{H}), \in^*)$ be the Polish relational system where

   $$\mathcal{S}(\omega, \mathcal{H}) := \{\varphi : \omega \to [\omega]^{<\omega} : \exists h \in \mathcal{H} \forall i < \omega (|\varphi(i)| \leq h(i))\},$$

   and $x \in^* \varphi$ if and only if $\exists n < \omega \forall i \geq n (x(i) \in \varphi(i))$, which is read *$x$ is localized by φ*. Bartoszyński proved that (see [BJ95, Theorem 2.3.9]), $\mathfrak{b}(\mathrm{LOC}) = \mathrm{add}(\mathcal{N})$ and $\mathfrak{d}(\mathrm{LOC}) = \mathrm{cof}(\mathcal{N})$.

   Moreover, any ν-centered poset is θ-LOC-good for any $\nu < \theta$ infinite (see [JS90]). In particular, σ-centered posets are LOC-good. Also subalgebras (not necessarily complete) of random forcing are LOC-good as a consequence of a result of Kamburelis [Kam89].

The following are the main general results concerning the preservation theory presented so far.

Lemma 0.25. Let $(\mathbb{P}_\alpha : \alpha < \theta)$ be a $\lessdot$-increasing sequence (see Definition 3.4) of ccc forcings and let $\mathbb{P}_\theta = \mathrm{limdir}_{\alpha<\theta} \mathbb{P}_\alpha$. If $\mathbb{P}_{\alpha+1}$ adds a Cohen real $\dot{c}_\alpha$ over $V^{\mathbb{P}_\alpha}$ for any $\alpha < \theta$, then $\mathbb{P}_\theta$ forces that $\{\dot{c}_\alpha : \alpha < \theta\}$ is a θ-$\mathcal{R}$-unbounded family of size θ.

Theorem 0.26. *Let $\delta \geq \theta$ be an ordinal and $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta)$ be a finite support iteration of non-trivial θ-$\mathcal{R}$-good ccc posets. Then, $\mathbb{P}_\delta$ forces $\mathfrak{b}(\mathcal{R}) \leq \theta$ and $\mathfrak{d}(\mathcal{R}) \geq |\delta|$.*

Proof. Since all finite support iterations of non-trivial forcings add Cohen reals at steps of countable cofinality, we know that at step θ we already have a family of Cohen reals of this size in $\mathbb{P}_\theta$. This family is θ-$\mathcal{R}$-unbounded because of the lemma above, and using the θ-$\mathcal{R}$-goodness of our posets, it will be preserved to be $\mathcal{R}$-unbounded until the last step of the iteration, that is, in $V^{\mathbb{P}_\delta}$ this family is $\mathcal{R}$-unbounded and so, $\mathfrak{b}(\mathcal{R}) \leq \theta$.

Moreover, if $\lambda \in [\theta, |\delta|)$ is a regular cardinal and $\mathcal{D}$ is a family of reals in the final extension of size λ note that it cannot be $\mathcal{R}$-dominating because at step $\mathbb{P}_\lambda$ we have added a family of λ-Cohen reals. Hence, the λ^+-th Cohen is not $\mathcal{R}$-dominated by any member of $\mathcal{D}$ and so $\mathfrak{d}(\mathcal{R}) \geq |\delta|$. □

0.4. Large cardinals

In various results, we will need to strengthen the kind of uncountable cardinals we are working with. That is why we present the definition of some large cardinal properties that we will use in various results.

Definition 0.27. Let κ be a cardinal number, we say that κ is:

1. a strong limit cardinal if $2^\lambda < \kappa$ for every $\lambda < \kappa$.
2. inaccessible, if it is regular, uncountable and a strong limit.
3. weakly compact if it is regular and for every partition $F : [\kappa]^2 \to 2$ there is a homogeneous set of size κ.
4. measurable, if there is a non-principal κ-complete ultrafilter $\mathcal{U}$ on κ ($\mathcal{U}$ is κ-complete if for all $\gamma < \kappa$ and $(A_\alpha : \alpha < \gamma) \subseteq \mathcal{U}$, the intersection $\bigcap_{\alpha<\gamma} A_\alpha \in \mathcal{U}$). Many times we will use that indeed κ is measurable if and only if there is a normal ultrafilter on κ, meaning that it is closed under diagonal intersections of size κ, i.e. if $(A_\alpha : \alpha < \kappa) \subseteq \mathcal{U}$ then $\triangle_{\alpha<\kappa} A_\alpha = \{\beta < \kappa : \beta \in \bigcap_{\alpha<\beta} A_\alpha\} \in \mathcal{U}$.
5. strongly compact if every κ-complete filter can be extended to a κ-complete ultrafilter.
6. λ-supercompact for $\lambda > \kappa$ if there is an elementary embedding $j : V \to M$ such that $j(\gamma) = \gamma$ when $\gamma < \kappa$; $j(\kappa) > \lambda$ and $M^\lambda \subseteq M$, i.e. every sequence $(a_\alpha : \alpha < \lambda)$ of elements of M belongs to M.
7. supercompact if it is λ-supercompact for all $\lambda > \kappa$.

For properties of these cardinals we refer the reader to the classical literature, for instance, Jech's [Jec03] and Kanamori's [Kan80] books.

Part I.

Cardinal invariants on the uncountable

Chapter 1

Cichoń's Diagram on the uncountable

In this chapter, we present an attempt to a generalization of the well-known Cichoń's diagram to uncountable cardinals. Instead of restricting ourselves to the classic Cantor or Baire spaces (2^ω or ω^ω respectively), we will work on the space 2^κ or κ^κ when κ is a regular uncountable cardinal (sometimes even a large cardinal). Most of the combinatorial cardinal invariants involved in the classical diagram can easily be redefined in this context. Moreover, the meager ideal $\mathcal{M}$ on 2^ω has a natural analogue $\mathcal{M}_\kappa$ on κ^κ (or 2^κ) if we equip κ^κ (2^κ respectively) with the $<\kappa$-box topology.

Cardinals associated with the meager ideal together with the dominating and unbounding numbers have purely combinatorial descriptions (See Chapter 2 in [BJ95]) that can be easily lifted to the uncountable. On the other hand, cardinals associated with the ideal of null sets on ω^ω (or 2^ω) $\mathcal{N}$ have no straightforward generalizations, mainly because so far it is unclear how this ideal can be generalized to the uncountable (there is no obvious definition of measure for the spaces 2^κ or κ^κ). In order to obtain a version of Cichoń's diagram for uncountable regular κ containing at least some analogs of the cardinals related to $\mathcal{N}$, we generalize instead their existing combinatorial characterizations.

The first section of this chapter focuses on this part, i.e. how to define the cardinals and how to obtain the basic ZFC-inequalities between them. We show that when we assume κ to be strongly inaccessible we obtain a good approximation of the diagram in this context. The remaining sections study some consistency results involving the cardinal invariants of our new diagram: In the absence of the famous *preservation results* (see Chapter 6 in [BJ95]) our approach generalizes first some well-known iterations and products and then calculates the values of the cardinal invariants in the resulting extensions. Section 1.2 deals particularly with generic extensions obtained as $<\kappa$-support iterations of κ-centered forcing notions, while Section 1.3 studies models obtained as iterations and products with supports of size κ of the generalization of classic tree forcing notions.

1.1. ZFC results

Let κ be an uncountable regular cardinal satisfying $\kappa^{<\kappa} = \kappa$. Endow the space of functions κ^κ with the topology generated by the sets of the form: $[s] = \{f \in \kappa^\kappa : f \supseteq s\}$

for $s \in \kappa^{<\kappa}$.

Let NWD_κ be the collection of nowhere dense subsets of $\kappa^{<\kappa}$ with respect to this topology, recall that a set $A \subseteq \kappa^\kappa$ is *nowhere dense* if for every $s \in \kappa^{<\kappa}$ there exists $t \supseteq s$ such that $[t] \cap A = \emptyset$. We define then the generalized κ-*meager* sets in κ^κ to be κ-unions of elements in NWD_κ and denote $\mathcal{M}_\kappa$ to be the κ-ideal that κ-meager sets determine (here κ-ideal means an ideal that in addition is closed under unions of size $\leq \kappa$). It is well known that the Baire category theorem can be lifted to this context, i.e. it holds that the intersection of κ-many open dense sets is open (see [FHK14]).

Now we start lifting the classical definitions of the cardinals in the diagram. As we already pointed out, most of the generalizations are straightforward, yet we include them all for sake of completeness.

Definition 1.1. If f, g are functions in κ^κ, we say that $f <^* g$, if there exists an $\alpha < \kappa$ such that for all $\beta > \alpha$, $f(\beta) < g(\beta)$. In this case, we say that g *eventually dominates* f.

Definition 1.2. Let $\mathfrak{F}$ be a family of functions from κ to κ.

- $\mathfrak{F}$ is *dominating*, if for all $g \in \kappa^\kappa$, there exists an $f \in \mathfrak{F}$ such that $g <^* f$.
- $\mathfrak{F}$ is *unbounded*, if for all $g \in \kappa^\kappa$, there exists an $f \in \mathfrak{F}$ such that $f \not<^* g$.

Definition 1.3 (The unbounding and dominating numbers, $\mathfrak{b}(\kappa)$ and $\mathfrak{d}(\kappa)$)**.**

- $\mathfrak{b}(\kappa) = \min\{|\mathfrak{F}|\colon \mathfrak{F}$ is an unbounded family of functions in $\kappa^\kappa\}$.
- $\mathfrak{d}(\kappa) = \min\{|\mathfrak{F}|\colon \mathfrak{F}$ is a dominating family of functions in $\kappa^\kappa\}$.

Definition 1.4 (Cardinal invariants associated to an ideal)**.** Let $\mathcal{I}$ be a κ-ideal on a set X:

- The additivity number:
$$\mathrm{add}(\mathcal{I}) = \min\{|\mathcal{J}|\colon \mathcal{J} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{J} \notin \mathcal{I}\}.$$
- The covering number:
$$\mathrm{cov}(\mathcal{I}) = \min\{|\mathcal{J}|\colon \mathcal{J} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{J} = X\}.$$
- The cofinality number:
$$\mathrm{cof}(\mathcal{I}) = \min\{|\mathcal{J}|\colon \mathcal{J} \subseteq \mathcal{I} \text{ and for all } M \in \mathcal{I} \text{ there is a } J \in \mathcal{J} \text{ with } M \subseteq J\}.$$
- The uniformity number:
$$\mathrm{non}(\mathcal{I}) = \min\{|Y|\colon Y \subset X \text{ and } Y \notin \mathcal{I}\}.$$

Once we have the definitions, our fist goal is to see what ZFC inequalities between them hold, having in account that our main motivation is to generalize the classical Cichoń's diagram (See Figure 1.1). The study of the generalization of the cardinals in the diagram started with the paper of Cummings and Shelah, *Cardinal invariants above the continuum* [CS95], where they studied both the dominating and unbounding numbers $\mathfrak{b}(\kappa)$ and $\mathfrak{d}(\kappa)$. The following is the first very basic result that establishes the ZFC relations between these two invariants.

$$\begin{tikzcd}
& \mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{N}) \arrow[r] & \mathfrak{c} \\
& & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & & \\
\aleph_1 \arrow[r] & \mathrm{add}(\mathcal{N}) \arrow[uu] \arrow[r] & \mathrm{add}(\mathcal{M}) \arrow[u] \arrow[r] & \mathrm{cov}(\mathcal{M}) \arrow[u] \arrow[r] & \mathrm{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}$$

Figure 1.1.: Classical Cichoń's diagram

Proposition 1.5 (See Theorem 1 in Cummings-Shelah [CS95])**.** Let κ be a regular uncountable cardinal, then:

1. $\kappa^+ \leq \mathfrak{b}(\kappa)$.
2. $\mathfrak{b}(\kappa) = \mathrm{cf}(\mathfrak{b}(\kappa))$.
3. $\mathfrak{b}(\kappa) \leq \mathrm{cf}(\mathfrak{d}(\kappa))$.
4. $\mathfrak{d}(\kappa) \leq 2^\kappa$.

Proof. Given $\{f_\alpha : \alpha < \kappa\} \subseteq \kappa^\kappa$, the function $g \in \kappa^\kappa$ defined by $g(\gamma) = \sup\{f_\beta(\gamma) : \beta \leq \gamma\}$ eventually dominates all the f_α's, so (1.) follows. For (2.) note that if $\mathcal{F}$ is an unbounded family of functions of size $\mathfrak{b}(\kappa)$, then we can write it as a union of $\mathrm{cf}(\mathfrak{b}(\kappa))$ many subfamilies $\mathcal{F}_\alpha$ of size $< \mathfrak{b}$. Thus there are $\mathrm{cf}(\mathfrak{b}(\kappa))$ many functions g_α, each one of them bounding the corresponding subfamily $\mathcal{F}_\alpha$, at the end the family $\{g_\alpha : \alpha < \mathrm{cf}(\mathfrak{b}(\kappa))\}$ is clearly unbounded. The proof of (3.) is analogous. $\square$

1.1.1. Cardinal invariants of the generalized meager ideal

Throughout this section we follow the approach from Blass in [Bla10] which uses Galois-Tukey connections to deduce inequalities between these cardinals.

Definition 1.6. A *κ-chopped function* is a pair (x, Π), where $x \in 2^\kappa$ and $\Pi = \{I_\alpha = [i_\alpha, i_{\alpha+1}) : \alpha < \kappa\}$ is an interval partition of κ. Here $i_0 = 0$ and the sequence $(i_\alpha : \alpha < \kappa)$ is strictly increasing and continuous. A function $y \in 2^\kappa$ *matches* a κ-chopped function (x, Π) if $x \upharpoonright I = y \upharpoonright I$ for unboundedly many intervals $I \in \Pi$. We denote from now on CR to be the set of κ-chopped functions on 2^κ.

In the countable case it is possible to characterize meagerness in terms of chopped reals, namely:

Theorem 1.7 (5.2 in [Bla10])**.** *A subset of 2^ω is meager if and only if there is a ω-chopped function that no member of M matches.*

Proof. The proof will appear later in a more general context. See 1.9 and 1.10 below. $\square$

In the paper *Mad families and their neighbors* [BHZ] Blass, Hyttinen, and Zhang studied the generalizations of some cardinals related to the meager ideal, one of their results shows that the characterization of meager sets presented above does not generalize to the uncountable.

Definition 1.8. We say that a subset M of 2^κ is *combinatorially meager* if there is a κ-chopped function that no member of M matches. We call $\mathrm{Match}(x, \Pi)$ the set of functions in 2^κ matching the κ-chopped function (x, Π).

Proposition 1.9 (Proposition 4.6 in [BHZ])**.** Every combinatorially meager set is meager.

Proof. Suppose that M is combinatorially meager and let (x, Π) the κ-chopped function witnessing it (here $\Pi = \{I_\alpha = [i_\alpha, i_{\alpha+1}) : \alpha < \kappa\}$), then the set $\mathrm{Match}(x, \Pi)$ can be written as follows:

$$\mathrm{Match}(x, \Pi) = \bigcap_{\alpha<\kappa} \bigcup_{\beta>\alpha} \{y \in 2^\kappa : y \restriction I_\beta = x \restriction I_\beta\}$$

which is clearly a comeager dense set in 2^κ. □

Proposition 1.10 (4.7 and 4.8 in [BHZ])**.** κ is strongly inaccessible if and only if every meager set is combinatorially meager.

Proof. We present just the direction from left to right because this is the only one we use in this document (to see the other one we refer to [BHZ]). Assume κ is strongly inaccessible and let M be meager, say $M = \bigcup_{\alpha<\kappa} F_\alpha$ where the F_α's form an increasing sequence of nowhere dense sets.

Recursively construct sequences $(i_\alpha : \alpha < \kappa) \subseteq \kappa$ and $(s_\alpha : \alpha < \kappa)$ such that:

- $(i_\alpha : \alpha < \kappa)$ is both increasing and continuous.
- $s_\alpha \in 2^{[i_\alpha, i_{\alpha+1})}$.
- $\forall t \in 2^{i_\alpha}$, $[t^\frown s_\alpha] \cap F_\alpha = \emptyset$.

The existence of this pair of sequences is guaranteed because of the inaccessibility of κ and the fact that the F_α's are nowhere dense sets. Finally, let x to be the concatenation of the s_α, that is, $x \restriction [i_\alpha, i_{\alpha+1}) = s_\alpha$ and I to be the interval partition determined by the i_α. Then no member of M matches (x, I), and so M is combinatorially meager. □

The following proposition will be useful for the upcoming results.

Proposition 1.11. $\mathrm{Match}(x,\Pi) \subseteq \mathrm{Match}(y,\Sigma)$ if and only if for all but $<\kappa$-many intervals $I \in \Pi$ there exists an interval $J \in \Sigma$ such that $J \subseteq I$ and $x \upharpoonright J = y \upharpoonright J$.

Proof. For the implication from left to right we argue by contradiction: If there are κ-many intervals $(I_\alpha : \alpha < \kappa) \subseteq \Pi$ such that, for all $J \in \Sigma$, $J \subseteq I_\alpha$ we have $x \upharpoonright J \neq y \upharpoonright J$. Then the function $x' \in \mathrm{Match}(x,\Pi) \setminus \mathrm{Match}(y,\Sigma)$ where $x' : \kappa \to 2$ is defined by:

$$x'(\alpha) = \begin{cases} x(\alpha) & \alpha \in I_\beta \text{ for some } \beta \\ 1 - y(\alpha) & \text{otherwise} \end{cases}$$

For the other direction, let $z \in \mathrm{Match}(x,\Pi)$, then there are κ-many intervals $(I_\alpha : \alpha < \kappa) \subseteq \Pi$ such that $z \upharpoonright I_\alpha = x \upharpoonright I_\alpha$, without loss of generality we can suppose that for all α there exists $J_\alpha \in \Sigma$ satisfying $J_\alpha \subseteq I_\alpha$, and so $z \upharpoonright J_\alpha = x \upharpoonright J_\alpha = y \upharpoonright J_\alpha$, thus $z \in \mathrm{Match}(y,\Sigma)$. □

Now we define and use the main results on Galois-Tukey connections (see [Bla10]), to characterize the cardinal invariants defined above and then obtain the desired inequalities.

Definition 1.12. Let $\mathbb{A} = (A_-, A_+, A)$ where, A_- and A_+ are two sets and A is a binary relation on $A_- \times A_+$. We define the *norm* of the triple $\mathbb{A}$, $\|\mathbb{A}\|$ as the smallest cardinality of any subset Y of A_+ such that every $x \in A_-$ is related by A to at least one element $y \in Y$. We also define the *dual* of $\mathbb{A}$, $\mathbb{A}^\perp = (A_+, A_-, \neg\check{A})$ where $(x,y) \in \neg\check{A}$ if and only if $(y,x) \notin A$.

Example 1.13. The cardinal invariants defined above can be seen as norms of some specific triples:

- If $\mathcal{D} = (\kappa^\kappa, \kappa^\kappa, \leq^*)$, then $\|\mathcal{D}\| = \mathfrak{d}(\kappa)$ and $\|\mathcal{D}^\perp\| = \mathfrak{b}(\kappa)$.
- If $\mathrm{Cov}(\mathcal{M}) = (2^\kappa, \mathcal{M}, \in)$, then $\|\mathrm{Cov}(\mathcal{M})\| = \mathrm{cov}\,\mathcal{M}(\kappa)$ and $\|\mathrm{Cov}(\mathcal{M})^\perp\|$ $= \mathrm{non}\,\mathcal{M}(\kappa)$.
- If $\mathrm{Cof}(\mathcal{M}) = (\mathcal{M}, \mathcal{M}, \subseteq)$, then $\|\mathrm{Cof}(\mathcal{M})\| = \mathrm{cof}\,\mathcal{M}(\kappa)$ and $\|\mathrm{Cof}(\mathcal{M})^\perp\|$ $= \mathrm{add}\,\mathcal{M}(\kappa)$.

Definition 1.14. A *morphism* between $\mathbb{A} = (A_-, A_+, A)$ and $\mathbb{B} = (B_-, B_+, B)$ (write $\Phi : \mathbb{A} \to \mathbb{B}$) is a pair $\Phi = (\Phi_-, \Phi_+)$ of maps satisfying:

- $\Phi_- : B_- \to A_-$.
- $\Phi_+ : A_+ \to B_+$.
- For all $b \in B_-$ and $a \in A_+$, if $\Phi_-(b) A a$ then $b B \Phi_+(a)$.

Proposition 1.15. If $\Phi : \mathbb{A} \to \mathbb{B}$, then $\|A\| \geq \|B\|$ and $\|A^\perp\| \leq \|B^\perp\|$.

Proof. Let $Y \subseteq A_+$ be such that for all $x \in A_-$, xAy for at least one $y \in Y$. Then if $x' \in B_-$, $x = \Phi(x') \in A_-$ and using the fact that Φ is a morphism we get $x' B \Phi_+(y)$ for at least one $y \in Y$, this clearly implies $\|A\| \geq \|B\|$. The dual case is analogous. □

Example 1.16. There are morphisms $\Phi : \mathrm{Cof}(\mathcal{M}) \to \mathrm{Cov}(\mathcal{M})$ and $\psi : \mathrm{Cof}(\mathcal{M}) \to \mathrm{Cov}(\mathcal{M})^{\perp}$ given by $\Phi = (S, id)$ and $\psi = (id, N)$ where $S(x) = \{x\}$ and for $M \in \mathcal{M}$, $N(M)$ is some arbitrary element of $2^\kappa \setminus M$.

Corollary 1.17.

- $\operatorname{add}\mathcal{M}(\kappa) \leq \operatorname{cov}\mathcal{M}(\kappa) \leq \operatorname{cof}\mathcal{M}(\kappa)$.
- $\operatorname{add}\mathcal{M}(\kappa) \leq \operatorname{non}\mathcal{M}(\kappa) \leq \operatorname{cof}\mathcal{M}(\kappa)$.

$$\begin{tikzcd}
 & \operatorname{non}\mathcal{M}(\kappa) \arrow[r] & \operatorname{cof}\mathcal{M}(\kappa) \arrow[r] & 2^\kappa \\
 & \mathfrak{b}(\kappa) \arrow[r] & \mathfrak{d}(\kappa) & \\
\kappa^+ \arrow[r] & \operatorname{add}\mathcal{M}(\kappa) \arrow[r] & \operatorname{cov}\mathcal{M}(\kappa) &
\end{tikzcd}$$

Figure 1.2.: Restriction of Cichoń's diagram for κ (I).

Summarizing we have the diagram shown in Figure 1.2. The inequality $\kappa^+ \leq \operatorname{add}\mathcal{M}(\kappa)$ follows from the fact that the union of κ-many, κ-meager sets is still κ-meager. In the upcoming results, the following characterization of the cardinals $\mathfrak{d}(\kappa)$ and $\mathfrak{b}(\kappa)$ will be used, this result is a simple generalization of Theorem 2.10 in [Bla10].

Definition 1.18. We say that an interval partition $I = \{I_\alpha : \alpha < \kappa\}$ *dominates* another interval partition $J = \{J_\alpha : \alpha < \kappa\}$ and write $J \leq^* I$ if $\exists\gamma\forall(\alpha > \gamma)\exists\beta(J_\beta \subseteq I_\alpha)$.

Theorem 1.19.

$$\mathfrak{d}(\kappa) = \min\{|\mathcal{I}| \colon \mathcal{I} \textit{ is a family of interval partitions and } \forall J \textit{ interval partition } \exists I \in \mathcal{I}(J \leq^* I)\}.$$

$$\mathfrak{b}(\kappa) = \min\{|\mathcal{I}| \colon \mathcal{I} \textit{ is a family of interval partitions such that } \nexists J \textit{ interval partition } \forall I \in \mathcal{I}(\mathcal{I} \leq^* \mathcal{J})\}.$$

Proof. The second item follows from the first item and duality. Suppose first that we have a family $\mathcal{I}$ of interval partitions dominating any other interval partition. To each one of the partitions $I = \{I_\alpha = [i_\alpha, i_{\alpha+1}) : \alpha < \kappa\}$ in $\mathcal{I}$, associate the function:

$$f_I(x) = i_{\alpha+2} \text{ where } \alpha \text{ is the unique ordinal} < \kappa \text{ such that } x \in I_\alpha$$

We want to show that $\{f_I : I \in \mathcal{I}\}$ is a dominating family: Given any increasing $g \in \kappa^\kappa$ form an interval partition satisfying:

$$\mathcal{J} = \{J_\alpha = [j_\alpha, j_{\alpha+1}] : \alpha < \kappa\} \text{ and whenever } x \leq j_\alpha \text{ then } g(x) < j_{\alpha+1}$$

Start with $j_0 = 0$, at successor steps put $j_{\alpha+1} = \sup\{g(x) : x \leq j_\alpha\}$ and in the limit steps take the supremum. Now let $I = \{I_\alpha = [i_\alpha, i_{\alpha+1} : \alpha < \kappa\}$ be an interval partition in $\mathcal{I}$ dominating $\mathcal{J}$, then we obtain $g(x) \leq f_I(x)$ for sufficiently large x. Notice that if α is the unique ordinal such that $x \in I_\alpha$, there exists β such that $J_\beta \subseteq I_{\alpha+1}$ (provided $\alpha > \gamma$ for a fixed γ given by the definition of dominating partitions).

Therefore, $i_\alpha \leq x < i_{\alpha+1}$, $x \leq j_\beta$ and by definition of g, $g(x) < j_{\beta+1} \leq i_{\alpha+2}$ and so $g(x) \leq f(x)$. We have proved $\mathfrak{d}(\kappa) \leq |\mathcal{I}|$.

For the other inequality, given a dominating family of functions $\mathcal{F}$ of size $\mathfrak{d}(\kappa)$ we can associate to each of them an interval partition I^g for $g \in \mathcal{F}$ as in the argument above. Then it is enough to prove that this set of interval partitions is dominating.

Let $J = \{J_\alpha = [j_\alpha, j_{\alpha+1}] : \alpha < \kappa\}$ and consider its associated function f_J, let also $g \in \mathcal{D}$ satisfying $f <^* g$, thus the partition I^g dominates J. Take then β sufficiently large such that $f(i_\beta) \leq g(i_\beta) \leq i_{\beta+1}$. But, by definition of f this means that the next J_γ after the one containing i_β lies entirely in I^g_β as we wanted. $\square$

Let IP denote the set of interval partitions and define the triple $\mathcal{D}' = (\mathrm{IP}, \mathrm{IP}, \leq^*)$. The theorem above allows us to conclude that $\|\mathcal{D}'\| = \mathfrak{d}(\kappa)$ and $\|(\mathcal{D}')^\perp\| = \mathfrak{b}(\kappa)$. The following two definitions were first used for Blass-Hyttinen and Zhang in [BHZ] to characterize the cardinals non $\mathcal{M}(\kappa)$ and cov $\mathcal{M}(\kappa)$ and are inspired on the classical combinatorial characterization of the countable versions of this cardinals by Bartoszyńsky [BJ95].

Definition 1.20. Let f and g be two functions in κ^κ. We say that f and g are *cofinally matching* if $|\{\alpha < \kappa : f(\alpha) = g(\alpha)\}| = \kappa$ (we write $f \bowtie g$). Otherwise we say f and g are *eventually different* (and write $f \neq^* g$). We define the following two cardinal invariants:

- $\mathrm{nm}(\kappa) = \min\{|\mathcal{F}|\colon (\forall g \in \kappa^\kappa)(\exists f \in \mathcal{F})(f \bowtie g)\}$.
- $\mathrm{cv}(\kappa) = \min\{|\mathcal{F}|\colon (\forall g \in \kappa^\kappa)(\exists f \in \mathcal{F})\neg(f \bowtie g)\}$.

In the countable case the cardinal invariants defined above coincide with non$(\mathcal{M})$ and cov$(\mathcal{M})$ respectively (see [BJ95], 2.4.A). In the uncountable we still have the following: Consider the triple $\mathcal{E} = (\kappa^\kappa, \kappa^\kappa, \bowtie)$ and note that $\|\mathcal{E}\| = \mathrm{cv}(\kappa)$ and $\|\mathcal{E}^\perp\| = \mathrm{nm}(\kappa)$, then:

Proposition 1.21.

- $\mathfrak{b}(\kappa) \leq \mathrm{nm}(\kappa) \leq \mathrm{non}\,\mathcal{M}(\kappa)$.
- $\mathrm{cov}\,\mathcal{M}(\kappa) \leq \mathrm{cv}(\kappa) \leq \mathfrak{d}(\kappa)$.

Proof. First we show that $\mathfrak{b}(\kappa) \leq \mathrm{nm}(\kappa)$ for which is enough to note that every cofinally matching family $\mathcal{F}$ is unbounded. Let $\mathcal{F}$ be cofinally matching and suppose that $\mathcal{F}$ is bounded, say by a function $g \in \kappa^\kappa$. Using that $\mathcal{F}$ is cofinally matching we can find a function $f \in \mathcal{F}$ such that $f \bowtie g$. But since $f <^* g$ there exists $\beta < \kappa$ such that, for all $\alpha > \beta$, $f(\alpha) < g(\alpha)$ which is a contradiction. Now, the inequality $\mathrm{cv}(\kappa) \geq \mathrm{cov}\,\mathcal{M}(\kappa)$ follows by observing that given $f \in \kappa^\kappa$ the set $X_f = \{g \in \kappa^\kappa : g \bowtie f\}$ is comeager, hence given $\{f_\alpha : \alpha < \mathrm{nm}(\kappa)\}$ is a witness for $\mathrm{cv}(\kappa)$ we have $\kappa^\kappa = \bigcup_{\alpha<\mathrm{nm}(\kappa)} X_{f_\alpha}$. The other inequalities follow from the duality properties of these cardinals.

$\square$

$$\begin{tikzcd}
 & \mathrm{non}\,\mathcal{M}(\kappa) \arrow[r] & \mathrm{cof}\,\mathcal{M}(\kappa) \arrow[r] & 2^\kappa \\
 & \mathfrak{b}(\kappa) \arrow[u] \arrow[r] & \mathfrak{d}(\kappa) & \\
\kappa^+ \arrow[r] & \mathrm{add}\,\mathcal{M}(\kappa) \arrow[r] & \mathrm{cov}\,\mathcal{M}(\kappa) \arrow[u] &
\end{tikzcd}$$

Figure 1.3.: Restriction of Cichoń's diagram for κ (II)

So far we have obtained the inequalities shown in Figure 1.3. Note also that up to this point we have just used that κ is a regular uncountable cardinal such that $\kappa^{<\kappa} = \kappa$. The following result is implicit in work of both Landver [Lan92] and Blass-Hyttinen-Zhang [BHZ] and shows that if we assume κ to be strongly inaccessible the uncountable case is analogous to the countable when characterizing the cardinals $\mathrm{non}\,\mathcal{M}(\kappa)$ and $\mathrm{cov}\,\mathcal{M}(\kappa)$. As usual, for cardinals λ, μ, and ν, we let $\mathrm{Fn}(\lambda, \mu, \nu)$ denote the set of partial functions from λ to μ with domain of size strictly less than ν.

Theorem 1.22. *Assume κ is strongly inaccessible. There are functions*

$$\Phi_- : \mathrm{CR} \times \mathrm{IP} \to ((\mathrm{Fn}(\kappa, 2, \kappa))^{<\kappa})^\kappa$$

and

$$\Phi_+ : \mathrm{IP} \times ((\mathrm{Fn}(\kappa, 2, \kappa))^{<\kappa})^\kappa \to 2^\kappa$$

such that if $(x, I) \in \mathrm{CR}$, $J \in \mathrm{IP}$, and $y \in ((\mathrm{Fn}(\kappa, 2, \kappa))^{<\kappa})^\kappa$ are such that if cofinally many J_α contain an interval of I and $\Phi_-((x, I), J)(\beta) = y(\beta)$ for cofinally many β, then $\Phi_+(J, y)$ matches (x, I).

Proof. Assume I and J are such that for cofinally many α, J_α contains an interval of I. Let $C = \{\alpha_\gamma : \gamma < \kappa\}$ be the enumeration of these α. That is, for $\gamma < \kappa$ there is $\delta_\gamma < \kappa$ such that $I_{\delta_\gamma} \subseteq J_{\alpha_\gamma}$. Put

$$\Phi_-((x, I), J)(\beta) = (x \restriction I_{\delta_\gamma} : \gamma < \omega_{\beta+1})$$

for $\beta < \kappa$. For other I and J, the value of $\Phi_-((x, I), J)(\beta)$ is arbitrary.

On the other hand $\Phi_+(J, y)$ is defined recursively. At each stage at most one interval J_α of J is considered and $\Phi_+(J, y) \restriction J_\alpha$ defined. Suppose we are at stage $\beta < \kappa$. If $y(\beta)$ is a sequence of length $\omega_{\beta+1}$ of partial functions all of whose domains are included in distinct J_α's, choose such J_α which has not been considered yet (this is possible by

$|\beta| \leq \omega_\beta < \omega_{\beta+1}$). Then let $\Phi_+(J,y) \restriction J_\alpha$ agree with the partial function from $y(\beta)$ whose domain is contained in J_α on its domain. If $y(\beta)$ is not of this form, do nothing. In the end, extend $\Phi_+(J,y)$ to a total function in 2^κ arbitrarily.

Now assume cofinally many J_α contain an interval of I and $\Phi_-((x,I),J)(\beta) = y(\beta)$ for cofinally many β. Fix such β. Then $y(\beta)$ is a sequence of length $\omega_{\beta+1}$ of partial functions all of whose domains are included in distinct J_α's and thus, for some γ, $\Phi_+(J,y) \restriction I_{\delta_\gamma}$ will agree with $x \restriction I_{\delta_\gamma}$. For different such β we must get distinct γ, and therefore $\Phi_+(J,y)$ matches (x,I). □

Corollary 1.23. Assume κ is strongly inaccessible.

1. (Blass, Hyttinen, Zhang [BHZ, 4.12 and 4.13]) $\mathrm{non}(\mathcal{M}_\kappa) = \mathrm{nm}(\kappa)$.
2. (Landver [Lan92]) $\mathrm{cov}(\mathcal{M}_\kappa) = \mathrm{cv}(\kappa)$.

Proof.

1. It suffices to prove $\mathrm{non}\,\mathcal{M}(\kappa) \leq \mathrm{nm}(\kappa)$ (see 1.21). Let $\mathcal{Y} \subseteq ((\mathrm{Fn}(\kappa,2,\kappa))^{<\kappa})^\kappa$ be a family of functions of size $\mathrm{nm}(\kappa)$ which is cofinally matching. Also let $\mathcal{J}$ be an unbounded family of partitions of size $\mathfrak{b}(\kappa) \leq \mathrm{nm}(\kappa)$.

   We claim $\{\Phi_+(J,y) : J \in \mathcal{J} \text{ and } y \in \mathcal{Y}\}$ is non-meager: if $(x,I) \in \mathrm{CR}$ and take $J \in \mathcal{J}$ to be unbounded over the partition given by taking unions of pairs of intervals of I, then cofinally many J_α contain an interval of I. If additionally $y \in \mathcal{Y}$ is such that $\Phi_-((x,I),J)(\beta) = y(\beta)$ for cofinally many β, then $\Phi_+(J,y)$ matches (x,I) and therefore does not belong to the meager set given by (x,I).
2. We should prove $\mathrm{cv}(\kappa) \leq \mathrm{cov}(\mathcal{M}_\kappa)$ (again see 1.21). Let $\mathcal{X} \subseteq \mathrm{CR}$ of size $< \mathrm{cv}(\kappa) \leq \mathfrak{d}(\kappa)$. First choose $J \in \mathrm{IP}$ such that cofinally many J_α contain an interval of I, for each I such that $(x,I) \in \mathcal{X}$ for some $x \in 2^\kappa$. Next choose $y \in ((\mathrm{Fn}(\kappa,2,\kappa))^{<\kappa})^\kappa$ such that for all $(x,I) \in \mathcal{X}$, $\Phi_-((x,I),J)(\beta) = y(\beta)$ for cofinally many β. Then $\Phi_+(J,y)$ matches (x,I) for all $(x,I) \in \mathcal{X}$, and therefore does not belong to any of the meager sets given by such (x,I).

□

Although along this chapter we deal mostly with the case κ strongly inaccessible, we would like to mention a couple of existing results in the other cases, among others to illustrate that in this case, the situation is rather different.

Theorem 1.24. *Assume κ is a successor cardinal.*

1. *(Hyttinen [Hyt06])* $\mathrm{nm}(\kappa) = \mathfrak{b}_\kappa$.
2. *(Matet, Shelah [MS12, Theorem 4.6]) If $2^{<\kappa} = \kappa$, then* $\mathrm{cv}(\kappa) = \mathfrak{d}(\kappa)$.

Note: It is still open whether it is consistent that κ is a successor cardinal and $\mathrm{cv}(\kappa) < \mathfrak{d}(\kappa)$.

Observation 1.25.

1. For $s \in 2^{<\kappa}$, $A_s = \{x \in 2^\kappa : \forall \alpha < \kappa\ (x \restriction [\alpha, \alpha + |s|) \neq s)\}$ is nowhere dense.
2. (Landver [Lan92, p. 1.3]) $2^{<\kappa} > \kappa$ implies $\operatorname{add} \mathcal{M}(\kappa) = \operatorname{cov} \mathcal{M}(\kappa) = \kappa^+$.
3. (Blass, Hyttinen, Zhang [BHZ, p. 4.15]) $\operatorname{non}(\mathcal{M}_\kappa) \geq 2^{<\kappa}$.

Proof. (1) is immediate. For (2), let $\lambda < \kappa$ be such that $2^\lambda > \kappa$. Then $2^\kappa = \bigcup\{A_s : s \in \Sigma\}$ for any $\Sigma \subseteq 2^\lambda$ with $|\Sigma| \geq \kappa^+$. Finally for (3), fix $X \subseteq 2^\kappa$ with $|X| < 2^{<\kappa}$. Let $\lambda < \kappa$ be such that $|X| < 2^\lambda$, hence $X \subseteq A_\sigma$ for some $\sigma \in 2^\lambda$. $\square$

There are just two inequalities missing in the diagram for which we will assume that κ is strongly inaccessible, such that we have that combinatorially meager sets and κ-meager sets coincide.

Definition 1.26. Let (x, Π) and (y, Σ) be two interval partitions. We say (x, Π) *engulfs* (y, Σ) if $\operatorname{Match}(x, \Pi) \subseteq \operatorname{Match}(y, \Sigma)$.

Then we have a morphism between the triples: $\operatorname{Cof}(\mathcal{M}) = (\mathcal{M}, \mathcal{M}, \subseteq)$ and $\operatorname{Cof}'(\mathcal{M}) = (\mathrm{CR}, \mathrm{CR}, \text{is engulfed by})$ where CR denote the set of κ-chopped functions. Define $\Phi_- : \mathrm{CR} \to \mathcal{M}$ by $\Phi_-(x, \Pi) = 2^\kappa \setminus \operatorname{Match}(x, \Pi)$ and $\Phi_+ : \mathrm{CR} \to \mathcal{M}$ by $\Phi_+(M) = (x_M, \Pi_M)$ to be one κ-chopped function that no member of M matches.

Thus $\|\operatorname{Cof}(\mathcal{M})\| = \|\operatorname{Cof}'(\mathcal{M})\| = \operatorname{cof} \mathcal{M}(\kappa)$ and $\|\operatorname{Cof}(\mathcal{M})^\perp\| = \|\operatorname{Cof}'(\mathcal{M})^\perp\| = \operatorname{add} \mathcal{M}(\kappa)$. In addition, note that if (x, Π) engulfs (y, Σ) then Π dominates Σ so $\|\mathcal{D}'\| \leq \|\operatorname{Cof}'(\mathcal{M})\|$ and $\|\operatorname{Cof}'(\mathcal{M})^\perp\| \leq \|\mathcal{D}'^\perp\|$.

In this way we obtain the inequalities $\mathfrak{d}(\kappa) \leq \operatorname{cof} \mathcal{M}(\kappa)$ and $\operatorname{add} \mathcal{M}(\kappa) \leq \mathfrak{b}(\kappa)$, and so the generalization of Cichoń's Diagram we were aiming for (see Figure 1.4). On the other hand, by Observation 1.25, $\operatorname{add} \mathcal{M}(\kappa) \leq \mathfrak{b}(\kappa)$ also holds when $2^{<\kappa} > \kappa$.

$$\begin{tikzcd}
 & \operatorname{non} \mathcal{M}(\kappa) \arrow[r] & \operatorname{cof} \mathcal{M}(\kappa) \arrow[r] & 2^\kappa \\
 & \mathfrak{b}(\kappa) \arrow[u] \arrow[r] & \mathfrak{d}(\kappa) \arrow[u] & \\
\kappa^+ \arrow[r] & \operatorname{add} \mathcal{M}(\kappa) \arrow[u] \arrow[r] & \operatorname{cov} \mathcal{M}(\kappa) \arrow[u] &
\end{tikzcd}$$

Figure 1.4.: Generalized Cichoń's diagram for κ strongly inaccessible

Finally, for κ strongly inaccessible it is also possible to obtain the two following important relationships (like in the ω case). Namely:

Proposition 1.27.

- $\mathrm{cof}\,\mathcal{M}(\kappa) = \max\{\mathrm{non}\,\mathcal{M}(\kappa), \mathfrak{d}(\kappa)\}$.
- $\mathrm{add}\,\mathcal{M}(\kappa) = \min\{\mathrm{cov}\,\mathcal{M}(\kappa), \mathfrak{b}(\kappa)\}$.

The proof will be a direct generalization of the one in [Bla10]; it uses the Galois-Tukey connections and duality.

Definition 1.28. Given two triples $\mathbb{A} = (A_-, A_+, A)$ and $\mathbb{B} = (B_-, B_+, B)$ we define the following operations:

- The categorical product $\mathbb{A} \times \mathbb{B}$ is $(A_- \dot{\cup} B_-, A_+ \times B_+, C)$ where $xC(a,b) \leftrightarrow xAa$ if $x \in A_-$ and xBb if $x \in B_-$.
- The conjunction $\mathbb{A} \wedge \mathbb{B}$ is $(A_- \times B_-, A_+ \times B_+, D)$ where $(x,y)D(a,b) \leftrightarrow xAa$ and yBb.
- The sequential composition $\mathbb{A};\mathbb{B}$ is $(A_- \times (B_-)^{A_+}, A_+ \times B_+, E)$ where $(x,f)E(a,b)$ if and only if xAa and $f(a)Bb$.

The dual operations are the categorical co-product $\mathbb{A} + \mathbb{B} = \mathbb{A}^\perp \times \mathbb{B}^\perp$, the disjunction $\mathbb{A} \vee \mathbb{B} = (\mathbb{A}^\perp \wedge \mathbb{B}^\perp)^\perp$ and the dual sequential composition $\mathbb{A} \!\stackrel{\cdot}{,}\! \mathbb{B} = (\mathbb{A}^\perp;\mathbb{B}^\perp)^\perp$. The following theorem establish the properties of the norms of this operations:

Theorem 1.29 (4.11 in [Bla10])**.**

1. $\|\mathbb{A} \times \mathbb{B}\| = \max\{\|\mathbb{A}\|, \|\mathbb{B}\|\}$.
2. $\|\mathbb{A};\mathbb{B}\| = \|\mathbb{A}\| \cdot \|\mathbb{B}\|$.
3. $\|\mathbb{A} + \mathbb{B}\| = \min\{\|\mathbb{A}\|, \|\mathbb{B}\|\}$.
4. $\|\mathbb{A} \vee \mathbb{B}\| = \min\{\|\mathbb{A}\|, \|\mathbb{B}\|\}$.
5. $\|\mathbb{A} \!\stackrel{\cdot}{,}\! \mathbb{B}\| = \min\{\|\mathbb{A}\|, \|\mathbb{B}\|\}$.

Proof of Proposition 1.27. It is clear that using duality it is enough to prove, for example $\mathrm{cof}\,\mathcal{M}(\kappa) = \max\{\mathrm{non}\,\mathcal{M}(\kappa), \mathfrak{d}(\kappa)\}$. The inequality $\mathrm{cof}\,\mathcal{M}(\kappa) \geq \max\{\mathrm{non}\,\mathcal{M}(\kappa), \mathfrak{d}(\kappa)\}$ is immediate. For the other we use the theorem above together with the construction of a morphism from the triple $((\mathrm{Cov}'(\mathcal{M}))^\perp; \mathcal{D}')$ to $\mathrm{Cof}'(\mathcal{M})$. This morphism is defined in the following form: $\Phi = (\Phi_-, \Phi_+)$ with $\Phi_- : \mathrm{CR} \to \mathrm{CR} \times \mathrm{IP}^{(2^\kappa)}$ and $\Phi_+ : 2^\kappa \times \mathrm{IP} \to \mathrm{CR}$. Specifically $\Phi_-((x,\Pi)) = (id, \varphi(x,\Phi))$, and φ is a function from CR to $\mathrm{IP}^{(2^\kappa)}$ defined as follows:

$$\varphi((x,\Phi)(y) = \begin{cases} \Sigma_y & \text{if } y \in \mathrm{Match}(x,\Pi) \\ \text{Any interval partition } \Gamma & \text{otherwise} \end{cases}$$

Here Σ_y is an interval partition with the property that each one of its intervals contains an interval of Π for the intervals where x and y agree, its existence is guaranteed because $y \in \mathrm{Match}(x,\Pi)$). Now it is clear from the definitions that Φ is a morphism and so we obtain the desired inequality. $\square$

More generally: Equip 2^κ with addition $+$ modulo 2. For $A \subseteq 2^\kappa$ and $y \in 2^\kappa$, let $A + y = \{x + y : x \in A\}$. If also $B \subseteq 2^\kappa$, put $A + B = \{x + y : x \in A \text{ and } y \in B\}$. Finally, identify $\sigma \in 2^{<\kappa}$ with the function in 2^κ which agrees with σ on its domain and takes value 0 elsewhere.

Proposition 1.30. Assume $2^{<\kappa} = \kappa$. There are functions $\Phi_+ : 2^\kappa \times \kappa^\kappa \to \mathcal{M}_\kappa$ and $\Phi_- : 2^\kappa \times \mathrm{NWD}_\kappa \to \kappa^\kappa$ such that $x \notin B + 2^{<\kappa}$ and $f \geq^* \Phi_-(x, B)$ imply $B \subseteq \Phi_+(x, f)$.

Proof. Let $\{\sigma_\beta : \beta < \kappa\}$ list $2^{<\kappa}$. Put

$$\Phi_+(x, f) = \bigcup_{\alpha<\kappa} \bigcap_{\beta\geq\alpha} 2^\kappa \setminus [(\sigma_\beta + x) \restriction f(\beta)].$$

This is clearly a meager set. For $x \notin B + 2^{<\kappa}$ let $\Phi_-(x, B)(\alpha)$ be such that $B \cap [(\sigma_\alpha + x) \restriction \Phi_-(x, B)(\alpha)] = \emptyset$. If $x \in B + 2^{<\kappa}$, define $\Phi_-(x, B)$ arbitrarily.

Now assume $x \notin B + 2^{<\kappa}$ and $f \geq^* \Phi_-(x, B)$. Let $y \in B$. Then $y \notin [(\sigma_\alpha + x) \restriction \Phi_-(x, B)(\alpha)]$ for all α. Since $f \geq^* \Phi_-(x, B)$, there is α such that $y \in 2^\kappa \setminus [(\sigma_\beta + x) \restriction f(\beta)]$ for all $\beta \geq \alpha$. Thus $y \in \Phi_+(x, f)$ as required. □

Corollary 1.31.

1. $\operatorname{add} \mathcal{M}(\kappa) \geq \min\{\mathfrak{b}(\kappa), \operatorname{cov} \mathcal{M}(\kappa)\}$.
2. Assume $2^{<\kappa} = \kappa$. Then $\operatorname{cof} \mathcal{M}(\kappa) \leq \max\{\mathfrak{d}(\kappa), \operatorname{non} \mathcal{M}(\kappa)\}$.

Proof.

1. If $2^{<\kappa} > \kappa$ this is immediate by Observation 1.25. If $2^{<\kappa} = \kappa$ use Proposition 1.30: If $\mathcal{B} \subseteq \mathrm{NWD}_\kappa$ with $|\mathcal{B}| < \min\{\mathfrak{b}(\kappa), \operatorname{cov} \mathcal{M}(\kappa)\}$, find $x \in 2^\kappa \setminus \bigcup \mathcal{B} + 2^{<\kappa}$ and then $f \in \kappa^\kappa$ with $f \geq^* \Phi_-(x, B)$ for all $B \in \mathcal{B}$. Then $\bigcup \mathcal{B} \subseteq \Phi_+(x, f)$.
2. Note that if $\mathcal{F} \subseteq \kappa^\kappa$ is dominating and $X \subseteq 2^\kappa$ is non-meager, then, by Proposition 1.30, $\{\Phi_+(x, f) : f \in \mathcal{F} \text{ and } x \in X\}$ is a cofinal family.

□

1.1.2. Slaloms and the invariants associated with them

The contents of this section are due to Andrew Brooke-Taylor and Jörg Brendle and are included in this document in order to give a complete presentation of the diagram for uncountable cardinals.

The classical Cichoń diagram contains cardinal invariants related to measure. While there are various attempts to generalize the ideal of null sets when κ is an inaccessible cardinal (see e.g. [She15] and [FL16]), we just consider generalizations of cardinal invariants which are combinatorial characterizations of the measure invariants in the countable case, similar to $\mathrm{nm}(\kappa)$ and $\mathrm{cv}(\kappa)$ for the category invariants. All through this section κ is always a – possibly weakly – inaccessible cardinal.

Definition 1.32 (Slaloms).

1. A function F with $\mathrm{dom}(F) = \kappa$ and $F(\alpha) \in [\kappa]^{|\alpha|}$ for $\alpha < \kappa$ is called a *slalom*. By $\mathrm{Loc}(\kappa)$ we denote the collection of all slaloms.
2. If $h \in \kappa^\kappa$ is a function with $\sup_{\alpha\to\kappa} h(\alpha) = \kappa$, and $F(\alpha) \in [\kappa]^{|h(\alpha)|}$ for all $\alpha < \kappa$, we say that F is an *h-slalom*. $\mathrm{Loc}_h(\kappa)$ is the set of h-slaloms. So $\mathrm{Loc}(\kappa) = \mathrm{Loc}_{\mathrm{id}}(\kappa)$.
3. A slalom F *captures* (or *localizes*) a function $f \in \kappa^\kappa$ ($f \in^* F$ in symbols) if $|\{\alpha < \kappa : f(\alpha) \notin F(\alpha)\}| < \kappa$.

Definition 1.33 (Cardinal invariants associated to slaloms).

- $\mathfrak{b}_h(\in^*)(\kappa) = \min\{|\mathcal{F}| \colon \mathcal{F} \subseteq \kappa^\kappa \text{ and } (\forall F \in \mathrm{Loc}_h(\kappa))(\exists f \in \mathcal{F})\neg(f \in^* F)\}$.
- $\mathfrak{d}_h(\in^*)(\kappa) = \min\{|\mathcal{F}| \colon \mathcal{F} \subseteq \mathrm{Loc}_h(\kappa) \text{ and } (\forall f \in \kappa^\kappa)(\exists F \in \mathcal{F})(f \in^* \varphi)\}$.

Thus the triple $\mathrm{LOC}_h(\kappa) = (\kappa^\kappa, \mathrm{Loc}_h(\kappa), \in^*)$ satisfies $||\mathrm{LOC}_h(\kappa)|| = \mathfrak{d}_h(\in^*)(\kappa)$ and $||\mathrm{LOC}_h(\kappa)^\perp|| = \mathfrak{b}_h(\in^*)(\kappa)$. In the countable case, these cardinals coincide with $\mathrm{add}(\mathcal{N})$ and $\mathrm{cof}(\mathcal{N})$ respectively for arbitrary functions $h \in \omega^\omega$ such that $\sup_{n\in\omega} h(n) = \omega$ [BJ95, p. 2.3]. Hence, all $\mathfrak{b}_h(\in^*)(\omega)$ are equal, and so are all $\mathfrak{d}_h(\in^*)(\omega)$. However in the uncountable this is consistently false (see 1.87).

Definition 1.34 (Partial slaloms).

1. Let $h \in \kappa^\kappa$ with $\sup_{\alpha\to\kappa} h(\alpha) = \kappa$. A function F is a *partial h-slalom* if $\mathrm{dom}(F) \subseteq \kappa$, $|\mathrm{dom}(F)| = \kappa$ and $F(\alpha) \in [\kappa]^{|h(\alpha)|}$ for $\alpha \in \mathrm{dom}(F)$. $\mathrm{pLoc}_h(\kappa)$ is the set of partial h-slaloms. For $h = \mathrm{id}$, write $\mathrm{pLoc}(\kappa) = \mathrm{pLoc}_{\mathrm{id}}(\kappa)$.
2. We say that F *localizes* $f \in \kappa^\kappa$ (write $f \in^* \varphi$) if $|\{\alpha \in \mathrm{dom}(\varphi) : f(\alpha) \notin \varphi(\alpha)\}| < \kappa$.

Analogously we define the associated cardinals:

Definition 1.35 (Cardinal invariants associated to partial slaloms).

- $\mathfrak{b}_h(\in^*_p)(\kappa) = \min\{|\mathcal{F}| \colon \mathcal{F} \subseteq \kappa^\kappa \text{ and } (\forall F \in \mathrm{pLoc}_h(\kappa))(\exists f \in \mathcal{F})\neg(f \in^* F)\}$.
- $\mathfrak{d}_h(\in^*_p)(\kappa) = \min\{|\mathcal{F}| \colon \mathcal{F} \subseteq \mathrm{pLoc}_h(\kappa) \text{ and } (\forall f \in \kappa^\kappa)(\exists \varphi \in \mathcal{F})(f \in^* \varphi)\}$.

If we define $\mathrm{pLOC}_h(\kappa) = (\kappa^\kappa, \mathrm{pLoc}_h(\kappa), \in^*)$, we see that $||\mathrm{pLOC}_h(\kappa)|| = \mathfrak{d}_h(\in^*_p)(\kappa)$ and $||\mathrm{pLOC}_h^\perp(\kappa)|| = \mathfrak{b}_h(\in^*_p)(\kappa)$.

Observation 1.36. Let κ be (weakly) inaccessible, then there are morphisms Φ : $\mathrm{LOC}_h(\kappa) \to \mathrm{pLOC}_h(\kappa)$ and $\Psi : \mathrm{pLOC}_h(\kappa) \to \mathcal{D}_\kappa$, so $\mathfrak{b}_h(\in^*)(\kappa) \leq \mathfrak{b}_h(\in^*_p)(\kappa) \leq \mathfrak{b}(\kappa)$ and $\mathfrak{d}(\kappa) \leq \mathfrak{d}_h(\in^*_p)(\kappa) \leq \mathfrak{d}_h(\in^*)(\kappa)$.

Proof. Just define $\Phi = (\Phi_- = \mathrm{id}_{\kappa^\kappa}, \Phi_+(F) = F) : F \in \mathrm{Loc}_h(\kappa)$ (every slalom is partial) and $\Psi = (\Psi_- : \mathrm{id}_{\kappa^\kappa}, \Psi_+(G)(\alpha) = \sup\{G(\alpha)\} : G \in \mathrm{pLoc}_h(\kappa))$. □

Formally, we could also have defined $\mathrm{LOC}(\kappa)$ and $\mathrm{pLOC}(\kappa)$ for successors. However, it is easy to see that in this case the following structures are isomorphic $\mathrm{LOC}(\kappa) \equiv \mathrm{pLOC}(\kappa) \equiv \mathcal{D}_\kappa$ so that the resulting cardinals are equal to $\mathfrak{b}_\kappa$ and $\mathfrak{d}_\kappa$, respectively, and thus not interesting. Specifically define the morphism $\Phi : \mathcal{D}_\kappa \to \mathrm{LOC}_h(\kappa)$ as $\Phi = (\Phi_- = \mathrm{id}_{\kappa^\kappa}, \Phi_+(f) = F_f : f \in \kappa^\kappa)$ where F_f is a slalom constructed inductively as follows: $F_f(0) = \{f(0)\}$, $F(\alpha+1) = F(\alpha) \cup \{f(\alpha+1)\}$ and $F(\gamma) = \bigcup_{\alpha<\gamma} F(\alpha)$ for γ a limit ordinal. Note that the fact that κ is a successor is crucial to guarantee that the function F_f is a slalom.

In the countable case, the partial localization cardinals do not depend on the function h [Bre95]. This is still true for (weakly) inaccessible κ.

Observation 1.37. Let g and h with $\sup_{\alpha\to\kappa} g(\alpha) = \sup_{\alpha\to\kappa} h(\alpha) = \kappa$. Then there is a morphism $\Psi : \mathrm{pLOC}_g(\kappa) \to \mathrm{pLOC}_h$.

Proof. Choose a strictly increasing sequence of ordinals $(\alpha_\gamma : \gamma < \kappa)$ with $h(\alpha_\gamma) \geq g(\gamma)$. Given $f \in \kappa^\kappa$, let $\Psi_-(f)(\gamma) = f(\alpha_\gamma)$ for all $\gamma < \kappa$. On the other hand, given $F \in \mathrm{pLoc}_g(\kappa)$, let $\mathrm{dom}(\Psi_+(F)) = \{\alpha_\gamma : \gamma \in \mathrm{dom}(F)\}$ and $\Psi_+(F)(\alpha_\gamma) = F(\gamma) \in [\kappa]^{|g(\gamma)|} \subseteq [\kappa]^{|h(\alpha_\gamma)|}$ for $\gamma \in \mathrm{dom}(F)$. Thus if $\Psi_-(f) \in^* F$, then $f \in^* \Psi_+(F)$ because $f(\alpha_\gamma) = \Psi_-(f)(\gamma) \in F(\gamma) = \Psi_+(F)(\alpha_\gamma)$ holds for all large enough $\gamma \in \mathrm{dom}(F)$. □

Corollary 1.38. Assume κ is (weakly) inaccessible. For any function $h \in \kappa^\kappa$ with $\sup_{\alpha\to\kappa} h(\alpha) = \kappa$, $\mathfrak{b}_h(\in^*_p)(\kappa) = \mathfrak{b}(\in^*_p)(\kappa)$ and $\mathfrak{d}_h(\in^*_p)(\kappa) = \mathfrak{d}(\in^*_p)(\kappa)$.

For the remainder of this subsection, we assume that κ is a strongly inaccessible cardinal. The next results show that in the strongly inaccessible case the classical Bartoszyński-Raisonnier-Stern Theorem [BJ95, Theorem 2.3.1] which asserts the existence of a morphism between $\mathrm{Cof}(\mathcal{N})$ and $\mathrm{Cof}(\mathcal{M})$ holds. This result is proved by constructing morphisms between $\mathrm{Cof}(\mathcal{N})$ and $\mathrm{LOC}(\omega)$ (in both directions) and $\mathrm{LOC}(\omega)$ and $\mathrm{Cof}(\mathcal{M})$. Moreover, a detailed analysis of the proof (see e.g. [Bre95, p. 2.5]) shows that there is a morphism between $\mathrm{pLOC}(\omega)$ and $\mathrm{Cof}(\mathcal{M})$.

Lemma 1.39 (Brendle–Brooke-Taylor)**.** Assume κ is strongly inaccessible. Let $X \subseteq 2^\kappa$ be a non-empty open set and let $\lambda < \kappa$, then there is a family $\mathcal{Y}$ of open subsets of X such that

1. $|\mathcal{Y}| \leq \kappa$,
2. every dense open subset of 2^κ contains a member of $\mathcal{Y}$,
3. $\bigcap \mathcal{Y}' \neq \emptyset$ for any $\mathcal{Y}' \subseteq \mathcal{Y}$ with $|\mathcal{Y}'| \leq \lambda$.

Theorem 1.40. *Assume κ is strongly inaccessible. Then there are functions $\Phi_- : \mathcal{M}_\kappa \to \kappa^\kappa$ and $\Phi_+ : \mathrm{pLoc}(\kappa) \to \mathcal{M}_\kappa$ such that $\Phi_-(A) \in^* F$ implies $A \subseteq \Phi_+(F)$ for $A \in \mathcal{M}_\kappa$ and $F \in \mathrm{pLoc}(\kappa)$.*

Proof. See Theorem 40 in [Bre+16]. □

Corollary 1.41. Assume κ is strongly inaccessible, then we have $\mathfrak{b}(\in^*_p)(\kappa) \leq \mathrm{add}\,\mathcal{M}(\kappa)$ and $\mathrm{cof}\,\mathcal{M}(\kappa) \leq \mathfrak{d}(\in^*_p)(\kappa)$.

The cardinals we have discussed in this section can be displayed in the following diagram.

$$\begin{tikzcd}
 & & & \mathrm{non}\,\mathcal{M}(\kappa) \arrow[r] & \mathrm{cof}\,\mathcal{M}(\kappa) \arrow[r] & \mathfrak{d}(\in^*_p)(\kappa) \arrow[r] & \mathfrak{d}(\in^*)(\kappa) \arrow[r] & 2^\kappa \\
 & & & \mathfrak{b}(\kappa) \arrow[u] \arrow[r] & \mathfrak{d}(\kappa) \arrow[u] & & & \\
\kappa^+ \arrow[r] & \mathfrak{b}(\in^*)(\kappa) \arrow[r] & \mathfrak{b}(\in^*_p)(\kappa) \arrow[r] & \mathrm{add}\,\mathcal{M}(\kappa) \arrow[u] \arrow[r] & \mathrm{cov}\,\mathcal{M}(\kappa) \arrow[u] & & &
\end{tikzcd}$$

Figure 1.5.: Cichoń's Diagram on the uncountable for κ strongly inaccessible.

1.2. Products and $<\kappa$-Support Iterations

From now on we work over forcing extensions of a ground model that we call V and we assume that κ is a strongly inaccessible cardinal so that we have the diagram in Figure 1.5. Unless stated otherwise, V is a model of ZFC that also satisfies GCH.

1.2.1. κ-Cohen forcing

Definition 1.42. The natural generalization of Cohen forcing is given by:

$$\mathbb{C}_\kappa = \{s : s \text{ is a function in } \kappa^{<\kappa}\}$$

ordered by reverse inclusion, i.e. $t \leq s$ if and only if $t \supseteq s$.

Note: Of course, it is possible to define $\mathbb{C}_\kappa$ over the generalized Baire space 2^κ and it is clear that the forcings are equivalent. Throughout this document we work without distinction with both versions of this poset.

This forcing generically adds a function $c_G \in \kappa^\kappa$ given by $c_G = \bigcup G$ where G is a $\mathbb{C}_\kappa$-generic filter. Since $2^{<\kappa} = \kappa$, it has size κ and so the κ^+-chain condition. Analogous to the case ω, it is possible to prove that κ-Cohen generic functions are unbounded over all ground model elements of κ^κ: simply note that for any ground model function $g \in \kappa^\kappa$ and $\alpha < \kappa$ the sets $D_{g,\alpha} = \{s \in \kappa^{<\kappa} : \exists\beta \geq \alpha(g(\beta) \leq s(\beta))\}$ are dense in $\mathbb{C}_\kappa$.

The following crucial property from Cohen reals can be partially generalized to our context, namely: Given the κ-Borel algebra $\mathcal{B}_\kappa$, i.e. the smallest κ-algebra that contains the open sets of 2^κ; a transitive model $M \models$ ZFC of size κ and N be a generic extension of M obtained after forcing with a κ-closed forcing notion $\mathbb{P}$. Then given $B \in \mathcal{B} \cap M$ we can find a code $r^B \in \kappa^\kappa$ that describes the way B has been constructed and gives us a well-defined version of B in N in the sense that $B^N \cap M = B^M$. From now on we will call the decoded version of B in the extension B^*.

The word *partially* in the paragraph above comes from the following fact: in the countable case, the set of Borel codes is Π^1_1, and so absolute between transitive models of ZFC. In contrast, when we generalize the Borel hierarchy for uncountable cardinals, we have absoluteness of $\Pi^1_1(\kappa)$-statements between certain forcing extensions. Namely:

Proposition 1.43 (Lemma 2.7 in [FKK16])**.** Let $\mathbb{P}$ be a κ-closed forcing notion. Then $\Sigma^1_1(\kappa)$-formulas are absolute between V and $V^{\mathbb{P}}$.

Now we work with an equivalent version of κ-Cohen forcing, the quotient algebra $\mathcal{B}_\kappa/\mathcal{M}_\kappa$. This version of the poset allows us to prove the following properties (which are are simple generalizations of the existing analogous results for ω).

Proposition 1.44. Let G be $\mathcal{B}_\kappa/\mathcal{M}_\kappa$-generic over V. Then there is a unique $c_G \in \kappa^\kappa \cap V[G]$ such that for all $B \in \mathcal{B}_\kappa$, $c_G \in B^*$ if and only if $[B] \in G$.

Proof. Work in the generic extension $V[G]$, and construct a sequence $(s_\alpha : \alpha < \kappa)$ such that $s_\alpha \subsetneq s_\beta$ when $\alpha < \beta < \kappa$ and $[s_\alpha] \in G$ for all $\alpha < \kappa$. Start with $s_0 = \emptyset$, given s_α choose a condition $[B] \leq [s_\alpha]$ and note that the set $\mathcal{D} = \{[B] \in \mathcal{B}_\kappa/\mathcal{M}_\kappa : \exists \beta [B] \leq [s_\alpha{}^\frown \beta]\}$ is dense below $[s_\alpha]$. Then there is some β such that $[s_\alpha{}^\frown \beta] \in G$ and we can choose then $s_{\alpha+1} = s_\alpha{}^\frown \beta$. For limit ordinals just take unions. At the end $x_G = \bigcup_{\alpha<\kappa} s_\alpha$ has the required property.

To see this, we work by induction on the κ-Borel hierarchy: For basic sets the property follows from the construction of x, for the complement case it is enough to notice that the set $\{[B], [\kappa^\kappa \setminus B]\}$ is a maximal antichain. Finally, for κ-unions note that given $(B^*_\alpha : \alpha < \kappa) \subseteq \mathcal{B}_\kappa$, the set $\{[\bigcap_{\alpha<\kappa} B_\alpha]\} \cup \{[\kappa^\kappa \setminus B_\alpha] : \alpha < \kappa\}$ is also a maximal antichain. □

Corollary 1.45. Let M be a transitive model of ZFC of size κ. Then for all $c \in \kappa^\kappa$ that belongs to a generic extension of M via a κ-closed forcing notion we have: If c is $\mathcal{B}_\kappa/\mathcal{M}_\kappa$-generic over M then $c \notin B^*$ for all $B \in (\mathcal{M}_\kappa)^M$.

Proof. Let $B \in (\mathcal{M}_\kappa)^M$, since $[(\kappa^\kappa \cap M) \setminus B] = [\kappa^\kappa \cap M]$ using the lemma above we have $c \in ((\kappa^\kappa \cap M) \setminus B)^* = (\kappa^\kappa \cap M) \setminus B^*$, so $x \notin B^*$. □

Now, we want to decide the cardinals in the diagram, so we notice that, as in the countable case it is possible to prove the following properties:

Lemma 1.46.

1. If $\dot{f}$ is a $\mathbb{C}_\kappa$- name for a function in κ^κ, there exists $g_f \in \kappa^\kappa \cap V$ such that, $\Vdash_{\mathbb{C}_\kappa} \dot{f} \bowtie g_f$.
2. If $\dot{f}$ is a $\mathbb{C}_\kappa$- name for a function in κ^κ, there exists $g_f \in \kappa^\kappa \cap V$ such that for all $h \in \kappa^\kappa$, if $h \not\leq^* g$, then $\Vdash_{\mathbb{C}_\kappa} h \not\leq^* \dot{f}$.

Proof. For (1) we use that $|\mathbb{C}_\kappa| = \kappa$. Let $\mathbb{C}_\kappa = \{s_\alpha : \alpha < \kappa\}$ be an enumeration of it and for each $\alpha, \beta < \kappa$, define the ground model function $g_f(\alpha) = \min\{\delta < \kappa :$ for some extension t of s_α, $t \Vdash \dot{f}(\alpha) = \delta\}$. The proof is complete by showing that g_f works. Let $s \in \mathbb{C}_\kappa$ and $\beta < \kappa$, it is enough to find $t \leq s$ and $\alpha \geq \beta$ such that $t \Vdash \dot{f}(\alpha) = g_f(\alpha)$, but that is clear from the definition of g_f. The proof of (2) is completely analogous. □

Now let $V \models$ GCH and consider the product with support $<\kappa$ and length λ (here λ is a cardinal $> \kappa^+$ such that $\lambda^\kappa = \lambda$) of the κ-Cohen forcing $\mathbb{P} = \prod_{i\in\lambda} \mathbb{C}_\kappa$. The forcing $\mathbb{P}$ preserves cardinals thanks to the following result:

Proposition 1.47. If each $\mathbb{P}_i$ has size $\kappa > \omega$, then the λ-product of the $\mathbb{P}_i$ satisfies the κ^+-chain condition.

Proof. Let $\mathcal{A} \subseteq \mathbb{P}$ to be a maximal antichain and take $\mathcal{M}$ to be a elementary submodel of V of size κ that is closed under $<\kappa$ sequences and such that $\mathbb{P}, \mathcal{A} \in \mathcal{M}$. Consider $\mathcal{A} \cap \mathcal{M}$ which is by elementarity an antichain in $\mathcal{M}$ and clearly has size at most κ. Suppose $p \in \mathbb{P}$ is an arbitrary condition and take $p \cap \mathcal{M}$ which is an element of $\mathcal{M}$ and since $\operatorname{supp}(p)$ has size $<\kappa$ we can assume $\operatorname{supp}(p) = \operatorname{supp}(p \cap \mathcal{M})$. Finally $p \cap \mathcal{M}$ is compatible with some $q \in \mathcal{A} \cap \mathcal{M}$, but then it is clear that p is also compatible with q. □

We also have:

Lemma 1.48. Assume $\lambda \geq \kappa^+$ is a regular cardinal, and $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \lambda)$ is an iteration with supports of size $<\kappa$ such that all $\mathbb{P}_\alpha$ have the κ^+-cc for all $\alpha \leq \lambda$. Also, assume that $\mu < \lambda$ and $A \subseteq \mu$ belongs to the $\mathbb{P}_\lambda$-generic extension. Then there is $\alpha < \lambda$ such that A belongs to the $\mathbb{P}_\alpha$-generic extension.

Sketch. It is enough to notice that if $\dot{f}$ is a $\mathbb{P}$-name for a function from κ to κ and for each $\alpha < \kappa$, A_α is a maximal antichain deciding $\dot{f}(\alpha)$, since $\mathbb{P}$ is κ^+-cc and all the conditions in A_α have support of size $<\kappa$, the set $J = \bigcup\{\operatorname{dom}(s) : s \in \bigcup_{\alpha<\kappa} A_\alpha\}$ has size at most κ and $\dot{f}$ can be reconstructed as a $\mathbb{C}_\kappa^J = \prod_{i \in J} \mathbb{C}_\kappa$-name. □

Then if we consider the first κ^+, κ-Cohen functions added by $\mathbb{P}$ $\{c_\alpha : \alpha < \kappa^+\}$ we claim that this family is unbounded in $V^{\mathbb{P}}$ and so it witnesses $\mathfrak{b}(\kappa) = \kappa^+$. Suppose that there is a $\mathbb{P}$-name $\dot{f}$ for a function in κ^κ that eventually dominates all the elements in $\{c_\alpha : \alpha < \kappa^+\}$, then $\dot{f}$ can be seen as a $\mathbb{P}_\alpha$-name for some $\alpha < \kappa^+$, but then the κ-Cohen function $c_{\alpha+1}$ satisfies $c_{\alpha+1} \not\leq^* f$ which is a contradiction.

On the other hand, since all the κ-Cohens added (κ^{++}-many) are unbounded the same arguments in the paragraph above give us $\mathfrak{d}(\kappa) \geq \kappa^{++}$ and because of the GCH in V we also obtain $2^\kappa \leq \kappa^{++}$.

The reason why $\operatorname{non} \mathcal{M}(\kappa) = \kappa^+$ is because $\mathbb{C}_\kappa$ does not add eventually different functions (see Lemma 1.48, Definition 1.20 and Proposition 1.21).

Hence, $\{c_\alpha : \alpha < \kappa^+\}$ witnesses $\operatorname{non} \mathcal{M}(\kappa) \leq \kappa^+$ in $V^{\mathbb{P}}$ and since we are adding κ^{++}-many κ-Cohen functions, proposition 1.45 we obtain $\operatorname{cov} \mathcal{M}(\kappa) \geq \kappa^{++}$. Summarizing we have the following:

Proposition 1.49. Let κ be inaccessible and let $\lambda > \kappa^+$ with $\lambda^\kappa = \lambda$. Then there is a generic extension in which $\operatorname{non} \mathcal{M}(\kappa) = \kappa^+ < \operatorname{cov} \mathcal{M}(\kappa) = \lambda$.

Moreover, it is possible to obtain a stronger result by weakening the large cardinal assumptions on κ:

Given a function $f : 2^{<\kappa} \to 2^{<\kappa}$ with the property $s \subseteq f(s)$ for all $s \in 2^{<\kappa}$ we can define the set $A_f = \{x \in 2^\kappa : f(s) \not\subseteq x \text{ for all } s \in 2^{<\kappa}\}$. It is clear that A_f is a nowhere dense subset of 2^κ (given s, the open set determined by the extension $f(s)$ avoids A_f).

$$\begin{tikzcd}
 & \mathrm{non}\,\mathcal{M}(\kappa) \arrow[r] & \mathrm{cof}\,\mathcal{M}(\kappa) \arrow[r] & 2^{\kappa} \\
 & \mathfrak{b}(\kappa) \arrow[u] \arrow[r] & \mathfrak{d}(\kappa) \arrow[u] & \\
\kappa^{+} \arrow[r] & \mathrm{add}\,\mathcal{M}(\kappa) \arrow[u] \arrow[r] & \mathrm{cov}\,\mathcal{M}(\kappa) \arrow[u] &
\end{tikzcd}$$

Figure 1.6.: Effect of the product of $\mathbb{C}_\kappa$

Reciprocally, given $F \in \mathrm{NWD}_\kappa$, there exists f such that $F \subseteq A_f$ (f simply choses of all s a extension t_s such that $[t_s] \cap F = \emptyset$).

Proposition 1.50. Let $\kappa = 2^{<\kappa}$ be regular uncountable. Also let $\lambda > \kappa^+$ with $\lambda^\kappa = \lambda$. Then, in the $\mathbb{P}$-generic extension, $\mathrm{non}\,\mathcal{M}(\kappa) = \kappa^+ < \mathrm{cov}\,\mathcal{M}(\kappa) = 2^\kappa = \lambda$ holds. Here $\mathbb{C}_\kappa^\lambda$ is the product of $\mathbb{C}_\kappa$ of length λ.

Proof. $2^\kappa = \lambda$ is well-known. Let $\dot{f} : 2^{<\kappa} \to 2^{<\kappa}$ be a $\mathbb{P}$-name for a function with $t \subseteq \dot{f}(t)$ for all $t \in 2^{<\kappa}$. By the κ^+-cc there is $B_{\dot{f}} \subseteq \lambda$ of size at most κ such that $\dot{f}$ is already added by $\mathbb{P} \restriction B_{\dot{f}}$ (see also Lemma 1.48). Furthermore, if $\beta \notin B_{\dot{f}}$, then $\dot{c}_\beta$ is forced not to belong to $A_{\dot{f}}$.

To see this, let $p \in \mathbb{P}$ and take $t \in 2^{<\kappa}$ such that $p(\beta) \subseteq t$. Strengthen $p \restriction B_{\dot{f}} \in \mathbb{P}_{B_{\dot{f}}}$ to $s_0 \in \mathbb{P}_{B_{\dot{f}}}$ such that $s_0 \Vdash \dot{f}(t) = t_0$. Then, extend $s(\beta)$ to $s_1 = t_0$. Define $q \leq p$ by $q \restriction B_{\dot{f}} = s_0$, $q \restriction \{\beta\} = s_1$, and $q \restriction (\lambda \setminus (B_{\dot{f}} \cup \{\beta\})) = p \restriction (\lambda \setminus (B_{\dot{f}} \cup \{\beta\}))$. Clearly q forces $\dot{c}_\beta \notin A_{\dot{f}}$.)

Now, if $\mu < \lambda$, and $\dot{f}_\gamma$, $\gamma < \mu$, are such names, then, for $\beta \in \lambda \setminus \bigcup_{\gamma<\mu} B_{\dot{f}_\gamma}$, $\dot{c}_\beta$ will witness that $\bigcup A_{\dot{f}_\gamma} \neq 2^\kappa$, and then $\mathrm{cov}\,\mathcal{M}(\kappa) \geq \lambda$ follows.

To see $\mathrm{non}\,\mathcal{M}(\kappa) \leq \kappa^+$, we show that $\dot{C} = \{\dot{c}_\beta : \beta < \kappa^+\}$ is a non-meager set in the generic extension. Indeed, if $\dot{f}_\gamma$, $\gamma < \kappa$, are names as before and $\beta \in \kappa^+ \setminus \bigcup_{\gamma<\kappa} B_{\dot{f}_\gamma}$, then $\dot{c}_\beta$ witnesses that $\dot{C}$ is not contained the union of the $A_{\dot{f}_\gamma}$. $\square$

1.2.2. κ-Eventually different forcing

Definition 1.51. The generalization of the eventually different forcing to κ, $\mathbb{E}_\kappa$ has the form:

$$\mathbb{E}_\kappa = \{(s, F) : s \in \kappa^\kappa \text{ and } F \in [\kappa^\kappa]^{<\kappa}\}$$

With the order given by: $(s, F) \leq (t, G) \leftrightarrow s \supseteq t$, $F \supseteq G$ and $\forall g \in G\, \forall \alpha(\mathrm{dom}(t) \leq \alpha < \mathrm{dom}(s) \to s(\alpha) \neq g(\alpha))$.

The forcing $\mathbb{E}_\kappa$ generically adds a function $e_G \in \kappa^\kappa$ given by $e_G = \bigcup\{s : \exists F \in [\kappa^\kappa]^{<\kappa}((s,F) \in G)\}$ where G is a $\mathbb{E}_\kappa$-generic filter.

Moreover $\mathbb{E}_\kappa$ is κ-centered: $\mathbb{E}_\kappa = \bigcup_{s\in\kappa^{<\kappa}} \mathbb{E}^s_\kappa$ where $\mathbb{E}^s_\kappa = \{(s,F) \in \mathbb{E}_\kappa : F \in [\kappa^\kappa]^{<\kappa}\}$ and given a family of conditions in $\mathbb{E}^s_\kappa$ of size $<\kappa$ they have a common extension lying in $\mathbb{E}^s_\kappa$ and this is possible for each $s \in \kappa^{<\kappa}$. In consequence, we have that $\mathbb{E}_\kappa$ is κ^+-cc, also notice that this forcing notion is κ-closed.

The generic function e_G has the property of being eventually different from all the ground model functions in κ^κ. We prove now that if κ is measurable after forcing with $\mathbb{E}_\kappa$ the set of ground model functions in κ^κ remains unbounded in the generic extension.

Lemma 1.52. Assume $V^{\mathbb{E}_\kappa} \models \kappa$ is a measurable cardinal. For any $\mathbb{E}_\kappa$-name $\dot{f}$ for a function from κ to κ, there exists $h \in \kappa^\kappa$ (in the ground model), such that for all $g \in \kappa^\kappa$, if $g \not<^* h$ then $\Vdash_{\mathbb{E}_\kappa} g \not<^* \dot{f}$.

Proof. Let $\mathcal{U}$ be a normal ultrafilter on κ in the extension given by $\mathbb{E}_\kappa$. Let $s \in \kappa^{<\kappa}$ and $\alpha < \kappa$ cardinal. Define the function $h_{s,\alpha} : \kappa \to \kappa^+$ as follows:

$$h_{s,\alpha}(i) = \begin{cases} \min\{j < \kappa\colon \forall F \subseteq \kappa^\kappa\ (|F|= \alpha \\ \qquad\qquad \to (s,F) \not\Vdash \dot{f}(i) > j)\} & \text{if it exists,} \\ \kappa & \text{otherwise.} \end{cases}$$

Claim 1.53. For all $s \in \kappa^{<\kappa}$ and for all $\alpha < \kappa$ cardinal, $\operatorname{ran}(h_{s,\alpha}) \subseteq \kappa$.

Proof. Suppose towards a contradiction that there exists $i < \kappa$ such that for all $j < \kappa$, $\exists F_j \subseteq \kappa^\kappa$, $|F_j|= \alpha$ and $(s,F_j) \Vdash \dot{f}(i) > j$. Put $F_j = \{f^l_j : l < \alpha\}$ and suppose without loss of generality that for limit l, $f^l_j = \sup_{i<l} f^i_j$. We will construct $D = \{\gamma_{j+1} : j < \kappa\}$, for $\delta < \alpha$, $A^\delta \subseteq D$ unbounded and $f^\delta : A^\delta \to \kappa$ such that:

$$\text{For all } 0 < \rho < \kappa \begin{cases} If\ \ \gamma_\rho \in A^\delta, & \text{then } \forall j > \rho\ \ f^\delta(\gamma_\rho) = f^\delta_{\gamma_j}(\rho) \\ If\ \ \gamma_\rho \in D \setminus A^\delta, & \text{then } \forall j > \rho\ \ f^\delta_{\gamma_j}(\rho) > j \end{cases}$$

The construction will be done by recursion defining an auxiliary sequence of subsets of κ, $\{C^\rho \in \mathcal{U} : \rho < \alpha\}$ as follows:

- $C^0 = \kappa$,
- In the successor step, let $\rho < \kappa$ and assume that we have already $C^\rho \in \mathcal{U}$ and $D_\rho = \{\gamma_{j+1} : j < \rho\}$. Also, suppose $A^\delta \cap D_\rho$ and $f^\delta \restriction_{A^\delta\cap D_\rho}$ have been defined for all $\delta < \alpha$ in such a way that $C^\rho \cap (\sup(D_\rho) + 1) = \emptyset$ and (1.2.2) are satisfied.

  Define $\gamma_{\rho+1} = \min(C^\rho)$ and perform a new inductive construction for $\delta < \alpha$ to construct sets $(B_\delta : \delta < \alpha) \subseteq \mathcal{U}$ (Our goal is to define $A^\delta \cap D_{\rho+1}$ where $D_{\rho+1} = D_\rho \cup \{\gamma_{\rho+1}\}$).

  Start with $B_0 = C^\rho \setminus \{\gamma_{\rho+1}\}$; in the successor step suppose that for $\beta \leq \delta$ we already have constructed $B_\beta \in \mathcal{U}$ and we want to define $B_{\delta+1}$. Consider the following partition of $F : [B_\delta]^2 \to 2$:

$$f(\{i,j\}) = \begin{cases} 1 & \text{if } f_j^\delta(\rho+1) = f_i^\delta(\rho+1) \\ 0 & \text{otherwise} \end{cases}$$

Then using that κ is measurable, we can find $X \subseteq B_\delta$, $X \in \mathcal{U}$ such that $F''[X]^2$ is constant. Thus we have the two following cases:

- If $F''[X]^2 = 1$ we put $A^{\delta+1} \cap (\gamma_{\rho+1}+1) = (A^{\delta+1} \cap D_{\rho+1}) \cup \{\gamma_{\rho+1}\}$ and define $f^\delta(\gamma_{\rho+1}) \in \kappa$ and $B_{\delta+1} = X \subseteq B_\delta$ such that $f_j^\delta(\rho+1) = f^{\delta+1}(\gamma_{\rho+1})$ for any $j \in B_{\delta+1}$.
- If $F''[X]^2 = 0$ put $A^{\delta+1} \cap (\gamma_{\rho+1}+1) = (A^{\delta+1} \cap D_{\rho+1})$ and choose $B_{\delta+1} \subseteq B_\delta$ in $\mathcal{U}$ such that the sequence $\{f_j^\delta(\rho)\}_{j \in B_{\delta+1}}$ is strictly above $\rho+2$.

In the limit steps just take intersections $B_\delta = \bigcap_{\beta<\delta} B_\beta$ (it is possible because of κ-completeness) and at the end put $C^{\rho+1} = B_\alpha$.

- For the limit case (ρ limit) suppose that for all $\beta < \rho$ we have $C^\beta \in \mathcal{U}$ and $D^* = \{\gamma_j : j < \rho\}$. Also assume that $A^\delta \cap D^*$ and $f^\delta \restriction_{A^\delta \cap D^*}$ have been defined satisfying (3). Put $C^\rho = \cap_{j<\rho} C^j$ and consider the following two cases:
  - If the $(\gamma_{j+1})_{j<\rho}$ has a cofinal sequence (in ρ) of elements in $A^\delta \restriction \gamma$, then put $A^\delta \cap (\gamma_{\rho+1}+1) = (A^\delta \cap D_{\rho+1}) \cup \{\gamma_{\rho+1}\}$.
  - If from some point on $\gamma_{j+1} \notin A^\delta$, just put $A^\delta \cap (\gamma_{\rho+1}+1) = A^\delta \cap D_{\rho+1}$.

For each $\delta < \alpha$ choose some $g^\delta \in \kappa^\kappa$ such that $g^\delta(\rho) = f^\delta(\gamma_\rho)$ for any $\rho < \kappa$ and $\gamma_\rho \in A^\delta$, also put $F' = \{g^\delta : \delta < \alpha\}$. Now find $(t, F'') \leq (s, F')$ in $\mathbb{E}_\kappa$ and $j_0 < \kappa$ such that $(t, F'') \Vdash \dot{f}(i) = j_0$.

Choose now $j > j_0$ above all the elements in the set $\{t(\beta) : \beta < |t|\} \cup \{|t|\}$, so we obtain that $(t, F_{\gamma_j} \cup F'')$ is a common extension of both (t, F'') and (s, F_{γ_j}). The fact that it is an extension of (t, F'') is immediate, for the other condition we must consider two cases:

- If $\gamma_\rho \in A^\delta$, $f^\delta(\gamma_j) = f^\delta(\gamma_\rho) = g_\delta(\rho) \neq t(\rho)$ because $(t, F'') \leq (s, F')$.
- If $\gamma_\rho \notin A^\delta$, $f^\delta(\gamma_j) > j > t(\rho)$.

Thus, $(t, F_{\gamma_j} \cup F'') \Vdash j_0 = \dot{f}(i) > \gamma_j \geq j$ which is a contradiction. □

Finally, take $h \in \kappa^\kappa$ dominating $\{h_{s,\alpha} : s \in \kappa^{<\kappa}, \alpha < \kappa\}$ and assume that $g \in \kappa^\kappa$ is not dominated by h. Then given a condition $(s, F) \in \mathbb{E}_\kappa$ with $\alpha = |F|$, find first $i_0 < \kappa$ such that for all $j > i_0$, $h_{s,\alpha}(j) \leq h(j)$ and then $i > i_0$ satisfying $h(i) < g(i)$ (exists because $g \not\leq^* h$). Thus for this i we have $g(i) > h_{s,\alpha}(i)$ and since $(s, F) \nVdash \dot{f} > h_{s,\alpha}(i)$ by definition of $h_{s,\alpha}$, there is $(t, F'') \leq (s, F)$ with $(t, F'') \Vdash \dot{f}(i) \leq h_{s,\alpha}(i) < g(i)$. □

The following result guarantees that if we iterate the forcing $\mathbb{E}_\kappa$ with $<\kappa$-support for λ-many steps we can preserve the property in the lemma above can be preserved in the steps of cofinality κ. For the steps of cofinality $> \kappa$ the κ^+-cc is enough.

Theorem 1.54. *Let $\mathcal{H} \subseteq \kappa^\kappa$ be such that for every $\mathcal{H}' \in [\mathcal{H}]^{\leq\kappa}$, there exists $h \in \mathcal{H}$ satisfying that $(h' \leq^* h)$ for all $h' \in \mathcal{H}'$. If $(\mathbb{P}_\gamma, \dot{\mathbb{Q}}_\gamma : \gamma \in \alpha)$ is an iteration with $<\kappa$-support,* $\mathrm{cf}(\alpha) = \kappa$ *such that $\forall\gamma \in \alpha$, $\mathbb{P}_\gamma = \mathbb{P}_\alpha \restriction \gamma$ satisfies the κ^+-cc, for a stationary set of $\gamma < \alpha$, $\mathbb{P}_\beta$ is a direct limit for $\beta < \alpha$ limit and $\Vdash_{\mathbb{P}_\gamma} \check{\mathcal{H}}$ is unbounded then $\Vdash_{\mathbb{P}_\alpha} \check{\mathcal{H}}$ is unbounded.*

Proof. First note that using Theorem 16.30 in [Jec03], we have that $\mathbb{P}_\alpha$ has the κ^+-cc, now suppose towards a contradiction that there exists $f \in V^{\mathbb{P}_\alpha}$ such that f dominates $\check{\mathcal{H}}$. Let $\dot{f}$ be a $\mathbb{P}_\alpha$-name for it and let $\{\beta_\delta\}_{\delta<\kappa}$ be a cofinal sequence in α.

Let f_δ be a function $\kappa^\kappa \cap V[G_{\beta_\delta}]$ where $G_{\beta_\delta} = G \cap \mathbb{P}_{\beta_\delta}$ with the following property:

$$\begin{gathered}\forall i < \kappa, f_\delta(i) = j \leftrightarrow j \text{ is the minimum ordinal such that} \\ \exists q \in \mathbb{P}_\alpha(q \restriction \beta_\delta \in G_{\beta_\delta}) \text{ and } q \Vdash_\alpha \dot{f}(i) = \check{j}\end{gathered}$$

Then for every $\delta < \kappa$ there exists a function $h_\delta : \kappa \to \kappa \in \mathcal{H}$ such that, $V[G_{\beta_\delta}] \models h_\delta \not<^* f_\delta$. Then, since $\mathbb{P}_\alpha$ is κ^+-cc there is $\mathcal{C} \in [\mathcal{H}]^\kappa \cap V$ such that $\{h_\delta : \delta < \kappa\} \subseteq \mathcal{C}$ and a function $h \in \mathcal{H} \cap V$ satisfying $\mathcal{C} <^* h$.

In particular for all δ, there exists γ_δ such that $\forall i \geq \gamma_\delta(h_\delta(i) \leq h(i))$. On the other hand, using the assumption we have that $V[G] \models \mathcal{H} <^* f$, which means that there is $p \in G$ and $\gamma < \kappa$ such that:

$$\forall i \geq \gamma, p \Vdash \check{h}(i) \leq \dot{f}(i).$$

Fix β_δ such that $\mathrm{supp}(p) \subseteq \beta_\delta$. Then, since $V[G_{\beta_\delta}] \models h_\delta \not<^* f_\delta$, $V[G_{\beta_\delta}] \models \exists i > \max\{\gamma_\delta, \gamma\}$ and there is a condition $p' \in G_{\beta_\delta}$ such that $p' \Vdash \dot{f}_\delta(i) < \check{h}(i)$, where $\dot{f}_\delta$ is a $\mathbb{P}_{\beta_\delta}$ for f_δ.

Using the definition of f_δ we obtain that there is $q \in \mathbb{P}_\alpha$ such that $q \restriction \beta_\delta \in G_{\beta_\delta}$ and $q \Vdash_\alpha \dot{f}(i) = \check{j}$. Finally, since $p \restriction \beta_\delta$, p' and $q \restriction \beta_\delta$ belong to G_{β_δ}, we can find a common extension q' and, in consequence obtain:

$$q' \Vdash \dot{f}_\delta(i) = \dot{f}(i) < \check{h}_\delta(i) \leq \check{h}(i) \leq \dot{f}(i).$$

which is a contradiction. □

Discussion: The results above are included in this document to illustrate one of the main obstructions that appear when trying to generalize iteration theorems. In the case of eventually different forcing, we are able to show that the property "*The ground model functions in κ^κ remain unbounded in the generic extension*" is preserved in the single step when iterating the forcing $\mathbb{E}$ and in the steps of cofinality $\geq \kappa$ along a $<\kappa$-support iteration. However, it is unclear how to get such preservation properties in steps of cofinality $<\kappa$, that is why unfortunately, we cannot decide cardinals in the generic extension obtained by iterating the generalized eventually different forcing with $<\kappa$-support.

In the countable case, the analogous forcing extension (i.e. a finite support iteration of $\mathbb{E}$ of length $\geq \aleph_2$) gives us a model in which $\mathfrak{b} < \mathrm{non}(\mathcal{M})$. In the uncountable case, it is still open whether the correspondent inequality can be proved consistent when κ is strongly inaccessible.

1.2.3. κ-Hechler forcing

Definition 1.55. The generalization of Hechler forcing to κ, $\mathbb{D}_\kappa$ has the form:

$$\mathbb{D}_\kappa = \{(s,f) : s \in \kappa^{<\kappa} \text{ and } f \in \kappa^\kappa\}$$

with the order given by $(s,f) \leq (t,g) \leftrightarrow s \supseteq t$, f dominates g everywhere (i.e. $\forall\alpha < \kappa$ $(g(\alpha) < f(\alpha))$) and $\forall\alpha(\mathrm{dom}(t) \leq \alpha < \mathrm{dom}(s) \rightarrow s(\alpha) \geq g(\alpha))$.

Generically, $\mathbb{D}_\kappa$ adds a function in κ^κ, $d_G = \bigcup\{s : \exists f \in \kappa^\kappa, (s,f) \in G\}$, where G is a $\mathbb{D}_\kappa$-generic filter. Clearly this function eventually dominates all elements in $\kappa^\kappa \cap V$: For all $h \in \kappa^\kappa \cap V$ there exists $\alpha < \kappa$ such that the set:

$$D_{h,\alpha} = \{(t,g) : \forall\beta\,\mathrm{sup}(\mathrm{dom}(t)) > \beta \geq \alpha)t(\beta) > h(\beta)\}$$

is dense in $\mathbb{D}_\kappa$. Given (s,f) put $\alpha = \mathrm{sup}(\mathrm{dom}(s))$ and define the element of $\kappa^{<\kappa}$, $t = s \cup \{(\alpha, \max\{f(\alpha), h(\alpha)\})\}$ and $g \supseteq t$ with $g(\delta) = f(\delta)$ for all $\delta > \mathrm{sup}(\mathrm{dom}(t))$. Then $(t,g) \leq (s,f)$ and $(t,g) \in D_{h,\alpha}$ and using genericity we have the desired property.

In a similar way it is possible to prove that $\mathbb{D}_\kappa$ adds a generic function $c_d \in 2^\kappa$ generic for $\mathbb{C}_\kappa$, the κ-Cohen forcing. Define $c_d(\alpha) = d(\alpha)\ (\mod (2))$ and take $D \subseteq \mathbb{C}_\kappa$ dense and (s,f) is a condition in $\mathbb{E}_\kappa$; then it is possible to find $(t,g) \leq (s,f)$ and $u \in D$ satisfying $(t,g) \Vdash u \subseteq \dot{d}_G$.

Finally $\mathbb{D}_\kappa$ is κ-centered, so it has the κ^+-cc and also it is κ-closed. Hence as in the countable case, if we iterate $\mathbb{D}_\kappa$ with length $\lambda \geq \kappa^+$ and $<\kappa$-support we are adding λ many dominating functions that witness $\mathfrak{b}(\kappa) \geq \lambda$ and since, simultaneously we are adding κ-Cohen functions we obtain $\mathrm{cov}\,\mathcal{M}(\kappa) \geq \lambda$. Hence using the relation $\mathrm{add}\,\mathcal{M}(\kappa) = \min\{\mathrm{cov}\,\mathcal{M}(\kappa), \mathfrak{b}(\kappa)\}$ we conclude the following:

Proposition 1.56. Let $\kappa = 2^{<\kappa}$ be inaccessible. Also let $\lambda > \kappa^+$ with $\lambda^\kappa = \lambda$. Then, in the $\mathbb{H}^\lambda_\kappa$-generic extension, $\mathrm{add}\,\mathcal{M}(\kappa) = 2^\kappa = \lambda$ holds.

$$\begin{tikzcd}
 & \mathrm{non}\,\mathcal{M}(\kappa) \arrow[r] & \mathrm{cof}\,\mathcal{M}(\kappa) \arrow[r] & 2^\kappa \\
 & \mathfrak{b}(\kappa) \arrow[u] \arrow[r] & \mathfrak{d}(\kappa) \arrow[u] & \\
\kappa^+ \arrow[r] & \mathrm{add}\,\mathcal{M}(\kappa) \arrow[u] \arrow[r] & \mathrm{cov}\,\mathcal{M}(\kappa) \arrow[u] &
\end{tikzcd}$$

Figure 1.7.: Effect of the $\mathbb{D}_\kappa$ iteration of length $\lambda > \kappa^+$ on the diagram

The following result uses Hechler forcing to create a model where $2^{<\kappa} > \kappa$ and the middle part of Cichoń's diagram is split horizontally into three levels.

Theorem 1.57 (Brendle–Brooke-Taylor). *Assume* GCH *and let κ be a regular uncountable cardinal. There is a cofinality-preserving generic extension in which* $\operatorname{add}\mathcal{M}(\kappa) = \operatorname{cov}\mathcal{M}(\kappa) = \kappa^+$, $\mathfrak{b}(\kappa) = \operatorname{nm}(\kappa) = \mathfrak{d}(\kappa) = \operatorname{cv}(\kappa) = \kappa^{++}$, *and* $\operatorname{non}\mathcal{M}(\kappa) = \operatorname{cof}\mathcal{M}(\kappa) = 2^\omega = 2^\kappa = \kappa^{+++}$.

In case κ is strongly inaccessible in the ground model – and weakly inaccessible in the extension – the consistency of $\operatorname{cov}\mathcal{M}(\kappa) < \operatorname{cv}(\kappa)$ answers a question of Matet and Shelah [MS12, Section 4] (see also [Kho+16, Question 3.8, part 2]).

Proof. First add κ^{++} many κ-Hechler functions in an iteration with supports of size $<\kappa$. By an earlier comment, $\operatorname{add}\mathcal{M}(\kappa) = \mathfrak{b}(\kappa) = \mathfrak{d}(\kappa) = \operatorname{cof}\mathcal{M}(\kappa) = \kappa^{++} = 2^\kappa$ holds in the generic extension. Then, all cardinals mentioned in the statement of the theorem will be equal to κ^{++}. Also, $2^{<\kappa} = \kappa$ still holds.

Assume $\kappa \geq \omega_2$. Partition κ into intervals J_α, $\alpha < \kappa$, such that each J_α has size at least ω_1 and $<\kappa$. Consider the space $\bar{\mathcal{X}}$ of functions $\bar{f}$ such that $\operatorname{dom}(\bar{f}) = \kappa$ and $\bar{f}(\alpha) \in \kappa^{J_\alpha}$ for all $\alpha < \kappa$. Since $|\kappa^{J_\alpha}| = \kappa^{<\kappa} = \kappa$, $\bar{\mathcal{X}}$ can be identified with κ^κ. In particular, since $\operatorname{cv}(\kappa) = \kappa^{++}$, we see that whenever $\bar{\mathcal{F}} \subseteq \bar{\mathcal{X}}$ is of size $\leq \kappa^+$, then there is $\bar{g} \in \bar{\mathcal{X}}$ such that for all $\bar{f} \in \bar{\mathcal{F}}$, $\bar{g}(\alpha) = \bar{f}(\alpha)$ holds for cofinally many $\alpha < \kappa$.

The case $\kappa = \omega_1$ is a little more complicated. We consider all possible interval partitions J such that all J_α are countable. Clearly, there are ω_2 of them. We then let $\bar{\mathcal{X}}^J$ as above.

Now add κ^{+++} Cohen reals. By Observation 1.25 we see that $\operatorname{add}\mathcal{M}(\kappa) = \operatorname{cov}\mathcal{M}(\kappa) = \kappa^+$ and $\operatorname{non}\mathcal{M}(\kappa) = \operatorname{cof}\mathcal{M}(\kappa) = 2^\omega = 2^\kappa = \kappa^{+++}$. Also, by the ccc-ness, every new function in κ^κ is bounded by a function from the intermediate extension. This means that $\mathfrak{b}_\kappa$ and $\mathfrak{d}_\kappa$ are preserved, and their values are still κ^{++} in the generic extension.

The main part of the argument is to show that $\operatorname{nm}(\kappa) = \operatorname{cv}(\kappa) = \kappa^{++}$ in the final extension. It suffices to prove $\operatorname{cv}(\kappa) \geq \kappa^{++}$ and $\operatorname{nm}(\kappa) \leq \kappa^{++}$. Work now in the intermediate extension.

For the former inequality, let $\dot{\mathcal{F}} = \{\dot{f}_\beta : \beta < \kappa^+\}$ be a family of names for functions in κ^κ. For each $\beta < \kappa^+$, recursively produce a function $f_\beta \in \kappa^\kappa$, an interval partition $I^\beta = (I^\beta_\alpha = [i^\beta_\alpha, i^\beta_{\alpha+1}) : \alpha < \kappa)$, and, for each $\alpha < \kappa$, maximal antichains $\{p^\beta_{\alpha,\gamma} : \gamma \in I^\beta_\alpha\}$ such that all I^β_α are countable and $p^\beta_{\alpha,\gamma} \Vdash \dot{f}_\beta(\gamma) = f_\beta(\gamma)$. This is clearly possible by the ccc.

If $\kappa \geq \omega_2$, simply let $\bar{f}_\beta$ be the function defined by $\bar{f}_\beta(\alpha) = f_\beta \restriction J_\alpha$ for all $\alpha < \kappa$. By the above, there is $\bar{g} \in \bar{\mathcal{X}}$ such that for all $\beta < \kappa^+$, $\bar{g}(\alpha) = \bar{f}_\beta(\alpha)$ holds for cofinally many $\alpha < \kappa$. Define $g \in \kappa^\kappa$ by $g(\gamma) = \bar{g}(\alpha)(\gamma)$ if $\gamma \in J_\alpha$, for $\alpha < \kappa$. We claim that g is forced to agree with all $\dot{f}_\beta$ cofinally often.

To see this, fix $\beta < \kappa^+$. Also fix some γ_0. Let $\alpha \geq \gamma_0$ be such that $\bar{g}(\alpha) = \bar{f}_\beta(\alpha)$. Notice that J_α contains one of the intervals $I^\beta_{\alpha'}$ because J_α is uncountable and the intervals of I^β are countable. Let p be an arbitrary condition. There is $\gamma \in I^\beta_{\alpha'}$ such that $p^\beta_{\alpha',\gamma}$ is compatible with p. Let q be a common extension. Clearly

$$q \Vdash \dot{f}_\beta(\gamma) = f_\beta(\gamma) = \bar{f}_\beta(\alpha)(\gamma) = \bar{g}(\alpha)(\gamma) = g(\gamma),$$

as required.

If $\kappa = \omega_1$, first choose an interval partition J dominating all the interval partitions I^β, $\beta < \kappa^+$. This is possible because $\mathfrak{b}(\kappa) = \kappa^{++}$, by Proposition 1.19. Then redo the argument of the preceding paragraph with this J and the space $\bar{\mathcal{X}}^J$.

The proof of $\mathrm{nm}(\kappa) \leq \kappa^{++}$ is simpler. Let $\mathcal{G}$ be κ^κ of the intermediate extension. Clearly $|\mathcal{G}| = \kappa^{++}$. It suffices to prove $\mathcal{G}$ is a witness for $\mathrm{nm}(\kappa)$ in the final extension. Let $\dot{f}$ be a name for a function in κ^κ. Again use the ccc to recursively produce a function $f \in \kappa^\kappa$, an interval partition $I = (I_\alpha : \alpha < \kappa)$, and, for each $\alpha < \kappa$, maximal antichains $\{p_{\alpha,\gamma} : \gamma \in I_\alpha\}$ such that all I_α are countable and $p_{\alpha,\gamma} \Vdash \dot{f}(\gamma) = f(\gamma)$. Clearly $f \in \mathcal{G}$ and f is forced to agree with $\dot{f}$ cofinally often. $\square$

$$\begin{tikzcd}
 & \mathrm{non}\,\mathcal{M}(\kappa) \arrow[r] & \mathrm{cof}\,\mathcal{M}(\kappa) \arrow[r] & 2^\kappa \\
 & \mathfrak{b}(\kappa) \arrow[u] \arrow[r] & \mathfrak{d}(\kappa) \arrow[u] & \\
\kappa^+ \arrow[r] & \mathrm{add}\,\mathcal{M}(\kappa) \arrow[u] \arrow[r] & \mathrm{cov}\,\mathcal{M}(\kappa) \arrow[u] &
\end{tikzcd}$$

Figure 1.8.: Effect of the model obtained in Theorem 1.57.

1.2.4. κ-Mathias forcing and κ-Laver forcing

If we want a suitable version of Mathias forcing in the uncountable that has nice properties (κ-closedness, for instance) the natural generalization of Mathias forcing will not work. Note that given a family of countably many unbounded subsets of κ, its intersection can be empty. Thus, in this section, we rather work with a generalization of Mathias-Příkrý forcing. Assume that κ is a measurable cardinal and that $\mathcal{U}$ is a normal measure on κ [1].

Definition 1.58. Define the generalized Mathias forcing with respect to the ultrafilter $\mathcal{U}$ as follows:

$$\mathbb{M}^\kappa_\mathcal{U} = \{(s, A) : s \in [\kappa]^{<\kappa} \text{ and } A \in \mathcal{U}\}$$

with the order given by $(t, B) \leq (s, A)$ if and only if $t \supseteq s$, $B \subseteq A$ and $t \setminus s \subseteq A$.

[1] For the definition of the forcing it is actually enough to start with a κ-complete filter $\mathcal{F}$. However, for our purposes, the use of a normal measure is more convenient.

Clearly, this forcing notion is κ-centered and so κ^+-cc. Also, since $\mathcal{U}$ is κ-complete we have that the forcing is κ-closed and moreover κ-directed closed. In the upcoming results, we will establish the relationship between Mathias forcing and the generalization of Laver forcing with respect to a normal ultrafilter. Those relations allow us to decide the values of the cardinal invariants in the diagram.

Definition 1.59 (Generalized Laver forcing)**.** Let $\mathcal{U}$ be a κ-complete ultrafilter on κ.

- An $\mathcal{U}$-Laver tree is a κ-closed tree $T \subseteq \kappa^{<\kappa}$ of increasing sequences with the property that $\forall s \in T(|s| \geq |\mathrm{stem}(T)| \rightarrow \mathrm{succ}_T(s) \in \mathcal{U})\}$.
- The generalized Laver Forcing $\mathbb{L}^{\kappa}_{\mathcal{U}}$ consists of all $\mathcal{U}$-Laver trees with order given by inclusion.

Proposition 1.60. Generalized Laver forcing generically adds a dominating function from κ to κ.

Proof. Let G be a $\mathbb{L}^{\kappa}_{\mathcal{U}}$-generic filter. The Laver generic function in κ^{κ}, l_G is defined as follows: $l_G = \cap\{[T] : T \in G\}$ where $[T]$ is the set of branches in T. To show that l_G is a dominating function it is enough to notice that, for all $f \in \kappa^{\kappa}$ the set $T_f = \{s \in T : \forall\alpha(|\mathrm{stem}(T)| \leq \alpha < |s|) \rightarrow s(\alpha) > f(\alpha)\}$ is also a condition in $\mathbb{L}^{\kappa}_{\mathcal{U}}$ and $T_f \leq T$. By genericity we conclude that $V[G] \models \forall f \in V \cap \kappa^{\kappa}(f <^* l_G)$. □

Lemma 1.61. If $\mathcal{U}$ is a normal ultrafilter on κ, then $\mathbb{M}^{\kappa}_{\mathcal{U}}$ and $\mathbb{L}^{\kappa}_{\mathcal{U}}$ are forcing equivalent.

Proof. The main point that we use in this proof is that for $\mathcal{U}$ a normal ultrafilter on κ we have the following "Ramsey "-like property: For all $f : [\kappa]^{<\omega} \rightarrow \gamma$ with $\gamma < \kappa$, there is a set in $\mathcal{U}$ homogeneous for f, i.e. there exists $B \in \mathcal{U}$ and $\rho < \gamma$ such that $f''[B]^{\kappa} = \rho$. Also, it is worth remembering that in the countable case if $\mathcal{U}$ is a Ramsey Ultrafilter $\mathbb{M}(\mathcal{U}) \simeq \mathbb{L}(\mathcal{U})$.

We want to define a dense embedding $\varphi : \mathbb{M}^{\kappa}_{\mathcal{U}} \rightarrow \mathbb{L}^{\kappa}_{\mathcal{U}}$; take (s, A) a condition in $\mathbb{M}^{\kappa}_{\mathcal{U}}$ and define the tree $T = T_{(s,A)}$ as follows:

- $\sigma = \mathrm{stem}(T)$ is the increasing enumeration of s.
- If we already have constructed $\tau \in T_{\alpha}$, with $\tau \supseteq \sigma$, then $\tau^\frown\langle\gamma\rangle \in T_{\alpha+1}$ if and only if $\gamma \in A$ and $\gamma \geq \sup\{\tau(\beta) : \beta < \alpha\}$.
- In the limit steps just ensure that $\tau \in T_{\alpha}$ if and only if $\tau \restriction \beta \in T_{\beta}$.

Note that T is a condition in $\mathbb{L}^{\kappa}_{\mathcal{U}}$. For the limit steps note that if $\tau \in T_{\alpha}$ for α limit the set $\mathrm{succ}_T(\tau) \supseteq \bigcap_{\beta<\alpha} \mathrm{succ}_T(\tau \restriction \beta)$.

Now consider the map $\varphi : (s, A) \rightarrow T_{(s,A)}$. It is clear that it preserves $\leq$, thus it is enough to prove that the trees of the form $T_{(s,A)}$ are dense in $\mathbb{L}^{\kappa}_{\mathcal{U}}$. For that, take an arbitrary $T \in \mathbb{L}^{\kappa}_{\mathcal{U}}$ and define:

$$f(\{\alpha, \beta\}) = \begin{cases} 1 & \text{if } \forall s \in T \text{ with } \alpha \geq \sup\{s(\gamma) : \gamma < |s|\} \\ & \qquad\qquad (\alpha \leq \beta \rightarrow \beta \in \mathrm{succ}_T(s)) \\ 0 & \text{otherwise} \end{cases}$$

Using the Ramsey-like property we can find a set $B \in \mathcal{U}$ homogeneous for f. The color of B cannot be 0 because T is a Laver tree. Now, knowing that $f''[B]^2 = \{1\}$ we can define $s = \mathrm{ran}(\mathrm{stem}(T))$ and $A = B \cap \mathrm{succ}_T(\mathrm{stem}(T))$ and conclude that $T_{(s,A)} \leq T$ as we wanted. □

Corollary 1.62. If $\mathcal{U}$ is a normal ultrafilter on κ then $\mathbb{M}^{\kappa}_{\mathcal{U}}$ always adds dominating functions.

Besides, if we want to iterate $\mathbb{M}^{\kappa}_{\mathcal{U}}$ with $<\kappa$-support for some ultrafilter $\mathcal{U}$, the first step is to guarantee that we actually can choose a normal measure to force with. In other words, we would like to preserve the measurability of κ after forcing with our version of Mathias forcing. The easiest approach to solve this problem is to assume that κ is supercompact, then one of its main properties is the existence of the well-known Laver preparation (see the Chapter2 for a detailed presentation of the construction), which makes the supercompactness of κ indestructible by subsequent forcing with κ-directed-closed partial orders, and so allows us to choose normal ultrafilters to iterate.

Note that $\mathbb{M}^{\kappa}_{\mathcal{U}}$ is κ-directed closed and use the Laver preparation S_κ to obtain V^{S_κ}, the *Laver prepared model*. Now run an iteration with $<\kappa$-support of Mathias Forcing with respect to a normal measure chosen arbitrarily in the corresponding extension. Namely, we have $(\mathbb{P}_\beta, \dot{\mathbb{Q}}_\alpha : \beta \leq \lambda, \alpha \leq \lambda)$ where $\lambda > \kappa^+$ and $V^{\mathbb{P}_\alpha} \models \kappa$ is supercompact, hence inside this model we can choose a normal measure $\mathcal{U}$ and define $\dot{\mathbb{Q}}_\alpha = \mathbb{M}^{\kappa}_{\mathcal{U}}$.

As a consequence, in $V^{\mathbb{P}_\lambda}$ we have added λ-many dominating reals which witness $\mathfrak{b}(\kappa) = \lambda^+$. On the other hand, since in a $<\kappa$-support iteration we add κ-Cohen functions in every step of cofinality κ (see Proposition below) we have that $\mathrm{cov}\,\mathcal{M}(\kappa) = \lambda$. At the end, we obtain that the effect of this iteration is the same that is obtained by adding λ-many κ-Hechler functions. (See Figure 1.7).

Proposition 1.63. Let δ be an ordinal with $\mathrm{cf}(\delta) = \kappa$ and let $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \delta, \beta < \delta)$ be a $<\kappa$-support iteration of κ-closed forcings such that $\Vdash_\alpha \dot{\mathbb{Q}}_\alpha$ is not the trivial forcing notion, for all $\alpha < \delta$. Then $\mathbb{P}_\delta$ adds a κ-Cohen function over all $V^{\mathbb{P}_\alpha}$ for all $\alpha < \delta$.

Proof. Let $(\gamma_\alpha : \alpha < \kappa)$ be a cofinal sequence on δ, for all $\alpha < \delta$, let $\dot{q}_\alpha$ be a $\mathbb{P}_\gamma$-name for a condition in $\dot{\mathbb{Q}}_{\gamma_\alpha}$ and define $c \in 2^\kappa$ as follows: $c(\alpha) = 1$ if and only if the interpretation of $\dot{q}_\alpha$ according to the generic filter, $\dot{q}_\alpha[G_\alpha]$ belongs to G_α where $G_\alpha = G \cap V^{\mathbb{P}_{\gamma_\alpha}}$ is $\mathbb{P}_{\gamma_\alpha}$-generic.

We prove now that c is κ-Cohen over all $V^{\mathbb{P}_{\gamma_\alpha}}$ for all $\alpha < \kappa$. Let $D \subseteq \mathbb{C}_\kappa \cap V$ dense and $p \in \mathbb{P}_\delta$, there exists $\alpha < \kappa$ such that $p \in \mathbb{P}_{\gamma_\alpha}$ and without loss of generality we can assume that for all $\beta < \alpha$ either $p \Vdash p(\beta) \leq \dot{q}_\beta$ or $\Vdash p(\beta) \perp \dot{q}_\beta$. Define then $s \in 2^\alpha$ as $s(\beta) = 0$ if and only if the first alternative holds, then it is clear that $p \Vdash s \subseteq \dot{c}$ and this gives us that c is κ-Cohen. □

1.2.5. Generalized localization forcing

For this subsection, assume that κ is strongly inaccessible. The definitions and results in this section are also due to Brendle and Brooke-Taylor.

Definition 1.64. The *generalized localization forcing* $\mathbb{LOC}_\kappa$ is defined as follows:

- conditions are of the form $p = (\sigma^p, \varphi^p) = (\sigma, \varphi)$ such that for some ordinal $\gamma = \gamma^p < \kappa$, $\mathrm{dom}(\sigma) = \gamma$, $\sigma(\alpha) \in [\kappa]^{|\alpha|}$ for all $\alpha \in \gamma$, $\mathrm{dom}(\varphi) = \kappa$, and $\varphi(\alpha) \in [\kappa]^{\leq|\gamma|}$ for $\alpha < \kappa$;
- the order is given by $q = (\sigma^q, \varphi^q) \leq p = (\sigma^p, \varphi^p)$ if σ^q end-extends σ^p (i.e., $\sigma^p \subseteq \sigma^q$), $\varphi^q(\alpha) \supseteq \varphi^p(\alpha)$ for all $\alpha \in \kappa$, and $\varphi^p(\alpha) \subseteq \sigma^q(\alpha)$ for all $\alpha \in \gamma^q \setminus \gamma^p$.

$\mathbb{LOC}_\kappa$ generically adds a slalom $F = F_G \in \mathrm{Loc}(\kappa)$ given by $F = \bigcup\{\sigma : \exists\varphi\ ((\sigma, \varphi) \in G)\}$, where G is a $\mathbb{LOC}_\kappa$-generic filter. Clearly F localizes all functions in $\kappa^\kappa \cap V$. See [BJ95, p. 106] for localization forcing $\mathbb{LOC}$ on ω.

Lemma 1.65. $\mathbb{LOC}_\kappa$ is κ^+-cc and $<\kappa$-closed.

Proposition 1.66 (Brendle–Brooke-Taylor)**.** Let κ be strongly inaccessible and let $\lambda > \kappa^+$ with $\lambda^\kappa = \lambda$. Then:

1. $\kappa^+ < \mathfrak{b}_{\mathrm{id}}(\in^*)(\kappa) = \lambda = 2^\kappa$ holds in a $<\kappa$-closed κ^+-cc extension.
2. $\kappa^+ = \mathfrak{d}_{\mathrm{id}}(\in^*)(\kappa) < 2^\kappa = \lambda$ holds in a $<\kappa$-closed κ^+-cc extension.

Proof. For (1) perform a λ-length iteration $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \lambda, \beta < \lambda)$ with $<\kappa$-support of $\mathbb{LOC}_\kappa$. The iteration still is $<\kappa$-closed and κ^+-cc. The argument for the latter is like for Hechler forcing, see Subsection 1.2.3. By Lemma 1.48 and genericity, we see that $2^\kappa = \mathfrak{b}_{\mathrm{id}}(\in^*)(\kappa) = \lambda$ in the resulting model.

For (2), assume $2^\kappa = \lambda$ in the ground model. Perform an iteration $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \kappa^+, \beta < \kappa^+)$ with $<\kappa$-support of $\mathbb{LOC}_\kappa$ of length κ^+. By Lemma 1.48 and genericity, we see that $\mathfrak{d}_{\mathrm{id}}(\in^*)(\kappa) = \kappa^+$ in the resulting model. □

The following result shows that the generalization of the Bartoszyński-Raisonnier-Stern Theorem (Theorem 1.40) may fail for weakly inaccessible κ.

Proposition 1.67 (Brendle–Brooke-Taylor)**.** Assume GCH and let κ be a strongly inaccessible cardinal. There is a cofinality-preserving generic extension in which $\mathrm{add}\,\mathcal{M}(\kappa) = \mathrm{cov}\,\mathcal{M}(\kappa) = \kappa^+$, $\mathfrak{b}_{\mathrm{id}}(\in^*)(\kappa) = \mathfrak{d}_{\mathrm{id}}(\in^*)(\kappa) = \kappa^{++}$, and $\mathrm{non}\,\mathcal{M}(\kappa) = \mathrm{cof}\,\mathcal{M}(\kappa) = 2^\omega = 2^\kappa = \kappa^{+++}$.

Proof. The argument is similar to the proof of Theorem 1.57. First add κ^{++} many $\mathbb{LOC}_\kappa$ generics in an iteration with supports of size $<\kappa$. By Proposition 1.66, $\mathfrak{b}_{\mathrm{id}}(\in^*)(\kappa) = \mathfrak{d}_{\mathrm{id}}(\in^*)(\kappa) = \kappa^{++} = 2^\kappa$ holds in the generic extension. Note that the family $\mathcal{F} \subseteq \mathrm{Loc}(\kappa)$ witnessing the value of $\mathfrak{d}_{\mathrm{id}}(\in^*)(\kappa)$ has the property that for all $\varphi : \kappa \to [\kappa]^\omega$ there is $F \in \mathcal{F}$ such that for $\varphi(\alpha) \subseteq F(\alpha)$ for all but less than κ many α.

Now add κ^{+++} Cohen reals. By Observation 1.25 we see that $\operatorname{add}\mathcal{M}(\kappa) = \operatorname{cov}\mathcal{M}(\kappa) = \kappa^+$ and $\operatorname{non}\mathcal{M}(\kappa) = \operatorname{cof}\mathcal{M}(\kappa) = 2^\omega = 2^\kappa = \kappa^{+++}$. Also, by the ccc-ness, for every new function f in κ^κ, there is a function $\varphi : \kappa \to [\kappa]^\omega$ in the intermediate extension such that $f(\alpha) \in \varphi(\alpha)$ for all $\alpha < \kappa$. By the previous paragraph, this easily entails that $\mathfrak{b}_{\mathrm{id}}(\in^*)(\kappa)$ and $\mathfrak{d}_{\mathrm{id}}(\in^*)(\kappa)$ are preserved, and their values are still κ^{++} in the generic extension. □

Total slaloms versus partial slaloms

Let κ be an uncountable regular cardinal.

Definition 1.68. Assume $\mathbb{P}$ is $<\kappa$-closed and κ-centered, say $\mathbb{P} = \bigcup_{\gamma<\kappa} P_\gamma$ where all P_γ are $<\kappa$-centered. Say that $\mathbb{P}$ is *κ-centered with canonical lower bounds* if there is a function $f = f^{\mathbb{P}} : \kappa^{<\kappa} \to \kappa$ such that whenever $\lambda < \kappa$ and $(p_\alpha : \alpha < \lambda)$ is a decreasing sequence with $p_\alpha \in P_{\gamma_\alpha}$, then there is $p \in P_\gamma$ with $p \leq p_\alpha$ for all $\alpha < \lambda$ and $\gamma = f(\gamma_\alpha : \alpha < \lambda)$.

Lemma 1.69. Let κ be an uncountable regular cardinal and assume $2^{<\kappa} = \kappa$. Let $\mu < (2^\kappa)^+$ be an ordinal. Assume $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \mu)$ is an iteration with $<\kappa$-support of $<\kappa$-closed, κ-centered forcing notions with canonical lower bounds such that the functions $f^{\dot{\mathbb{Q}}_\alpha}$ witnessing canonical lower bounds lie in the ground model. Then $\mathbb{P}_\mu$ is $<\kappa$-closed and κ-centered.

Definition 1.70. The *generalized partial localization forcing* $\mathbb{PLOC}_\kappa$ is defined as follows:

- conditions are of the form $p = (\sigma^p, \varphi^p) = (\sigma, \varphi)$ such that $\operatorname{dom}(\sigma) \subseteq \kappa$, $|\operatorname{dom}(\sigma)| < \kappa$, $\sigma(\alpha) \in [\kappa]^{|\alpha|}$ for all $\alpha \in \operatorname{dom}(\sigma)$, $\operatorname{dom}(\varphi) = \kappa$, and $\varphi(\alpha) \in [\kappa]^\lambda$ for all $\alpha < \kappa$, for some fixed $\lambda < \kappa$;
- the order is given by $q = (\sigma^q, \varphi^q) \leq p = (\sigma^p, \varphi^p)$ if σ^q end-extends σ^p (i.e., $\sigma^p \subseteq \sigma^q$ and $\alpha \in \operatorname{dom}(\sigma^q \setminus \sigma^p)$ implies $\alpha \geq \sup(\operatorname{dom}(\sigma^p))$), $\varphi^q(\alpha) \supseteq \varphi^p(\alpha)$ for all $\alpha \in \kappa$, and $\varphi^p(\alpha) \subseteq \sigma^q(\alpha)$ for all $\alpha \in \operatorname{dom}(\sigma^q \setminus \sigma^p)$.

$\mathbb{PLOC}_\kappa$ generically adds a partial slalom $F = F_G \in \operatorname{pLoc}(\kappa)$ given by $F = \bigcup\{\sigma : \exists\varphi\ ((\sigma, \varphi) \in G)\}$, where G is a $\mathbb{PLOC}_\kappa$-generic filter. Clearly F localizes all functions in $\kappa^\kappa \cap V$. See [Bre06, p. 47] for partial localization forcing $\mathbb{PLOC}$ on ω.

Lemma 1.71. Assume κ is strongly inaccessible. Then $\mathbb{PLOC}_\kappa$ is $<\kappa$-closed and κ-centered with canonical lower bounds. Furthermore, if $V \subseteq W$ are ZFC-models such that $2^{<\kappa} \cap V = 2^{<\kappa} \cap W$ and $\mathbb{PLOC} \in W$, then the function f witnessing canonical lower bounds may be taken in V.

Lemma 1.72. Let κ be strongly inaccessible and let $\mathbb{P}$ be a κ-centered forcing notion. Let $h \in \kappa^\kappa$ and assume $\dot{F}$ is a $\mathbb{P}$-name for an h-slalom. Then there are h-slaloms $(F_\alpha : \alpha < \kappa)$ such that whenever $f \in \kappa^\kappa$ is not localized by any F_α, then $\Vdash$ "$\dot{F}$ does not localize f".

Theorem 1.73 (Brendle–Broke-Taylor). *Let κ be strongly inaccessible and let $\lambda > \kappa^+$ with $\lambda^\kappa = \lambda$. Then:*

1. *$\kappa^+ = \mathfrak{b}_h(\in^*)(\kappa) < \mathfrak{b}_{\mathrm{id}}(\in^*_p)(\kappa) = \lambda = 2^\kappa$ (for all $h \in \kappa^\kappa$) holds in a $<\kappa$-closed κ^+-cc extension.*
2. *$\kappa^+ = \mathfrak{d}_{\mathrm{id}}(\in^*_p)(\kappa) < \mathfrak{d}_h(\in^*)(\kappa) = \lambda = 2^\kappa$ (for all $h \in \kappa^\kappa$) holds in a $<\kappa$-closed κ^+-cc extension.*

1.2.6. A note on the non-inaccessible case

In the ZFC section we observed that in the case when κ is a successor cardinal, there are two special sub-cases, namely when $2^{<\kappa} = \kappa$ and when $2^{<\kappa} > \kappa$. From them the values of certain cardinal invariants are completely determined (in the case of $\mathrm{nm}(\kappa)$) or bounded (like for $\mathrm{cv}(\kappa)$). This properties allows us to prove:

Observation 1.74. Let κ be a successor cardinal, then there is a cardinal preserving forcing extension such that:

1. $\mathfrak{d}(\kappa) = \kappa^+$ and $\operatorname{non}\mathcal{M}(\kappa) = \kappa^{++}$, and
2. $\operatorname{cov}\mathcal{M}(\kappa) = \kappa^+$ and $\mathfrak{b}(\kappa) = \kappa^{++}$.

Proof.

1. Start with a model $V \models \mathrm{GCH}$ and add κ^{++}-many Cohen reals (functions in 2^ω), then in the generic extension $\kappa^{++} = 2^\omega \leq 2^{<\kappa}$, then Observation 1.25 gives us $\operatorname{non}\mathcal{M}(\kappa) \geq 2^{<\kappa} > \kappa^+$ and since $\mathfrak{d}(\kappa) = \kappa^+$ and the forcing that adds this many Cohen reals is ccc, this will be preserved in the final extension.
2. Start with a model $V \models 2^{<\kappa} = \kappa$ and $\mathrm{GCH}(\kappa)$, now add κ^{++}-many κ-Hechler functions. Hence as we already saw in the previous section, in the resulting model we obtain $\mathfrak{b}(\kappa) = \kappa^{++} = 2^\kappa$. Add now κ^+ many Cohen reals. As in the proof above, the value of $\mathfrak{b}(\kappa)$ will be preserved (due to the ccc) and $\kappa^{++} = 2^\omega \leq 2^{<\kappa}$, so Observation 1.25 gives us $\operatorname{cov}\mathcal{M}(\kappa) = \kappa^+$.

□

It is not known whether one can prove the consistency of the inequalities above when κ is a successor such that $2^{<\kappa} = \kappa$.

1.3. κ-Support Iterations

Through the whole of this section we still assume that κ is at least a strongly inaccessible cardinal. In the countable case, countable support iterations of proper forcing notions provide plenty of models were cardinal invariants on Cichoń's Diagram can be separated. One of the main features that this kind of iterations has is that they do not necessarily add

Cohen reals and this is the desired behavior when one wants, for instance, to preserve the cardinal $\mathrm{cov}(\mathcal{M})$ small in the final extension.

The first challenge that these iterations entail is the preservation of the cardinal κ^+. In the countable case properness is the usual tool to guarantee this property, that in addition happens to be preserved under countable support iterations. In the uncountable, however, it is known that the straightforward generalization of properness for an uncountable cardinal fails to be preserved under κ-support iterations (See [Ros05] for an introductory discussion on this topic).

Additionally, one of the main tools to decide the values of the invariants in models obtained as countable support iterations are the well-known *preservation theorems*. In our case, this kind of theorems cannot be easily lifted, and so due to the lack of results in this direction, we use generalized fusion as a tool to preserve cardinals and to prove some properties of models obtained as κ-support iterations.

1.3.1. κ-Sacks forcing

The generalization of Sacks forcing for uncountable cardinals was first studied by Kanamori [Kan80]; we present his definitions and prove some properties of the iteration and the product of this forcing notion.

Recall that $T \subseteq 2^{<\kappa}$ is a *tree* if it is closed under initial segments, that is, $u \in T$ and $v \subseteq u$ imply $v \in T$. A node $u \in T$ *splits* in T if both $u^\frown 0$ and $u^\frown 1$ belong to T.

Definition 1.75. For strongly inaccessible κ, let $\mathbb{S}_\kappa$ be the following forcing notion: κ-*Sacks forcing* whose conditions are subtrees T of $2^{<\kappa}$ such that:

1. Each $u \in T$ has a splitting extension in $t \in T$, that is $u \subseteq t$ and t splits in T.
2. For any $\alpha < \kappa$, if $(u_\beta : \beta < \alpha)$ is a sequence of elements in T such that $\beta < \gamma < \alpha \to u_\beta \subseteq u_\gamma$, then $\bigcup\{u_\beta : \beta < \alpha\} \in T$.
3. If $\delta < \kappa$ is a limit ordinal, $u \in 2^\delta$ and for arbitrarily large $\beta < \delta$ if $u \restriction \beta$ splits in T, then u splits in T.

The extension relation is defined by $T \leq S$ if and only if $T \subseteq S$.

Note: It is clear that in the definition of the forcing one does not need to assume κ to be strongly inaccessible (it is enough κ regular with $2^{<\kappa} = \kappa$) Nonetheless, in order to have some useful properties of the forcing (like fusion, for instance) it will be clear why we are making this assumption.

As in the countable case we define the $\mathrm{stem}(T)$ where T is a condition in $\mathbb{S}_\kappa$ as the unique splitting node that is comparable with all elements in T. In addition, by recursion on κ we define:

Definition 1.76 (The α-th *splitting level* of T)**.** Given $T \in \mathbb{S}_\kappa$ define:

- $\mathrm{split}_0(T) = \mathrm{stem}(T)$.
- $\mathrm{split}_{\alpha+1}(T) = \{\mathrm{stem}(T_{u^\frown i}) : u \in \mathrm{split}_\alpha(T) \text{ and } i \in 2\}$.

- If δ is a limit ordinal $< \kappa$, we define $\text{split}_\delta(T) = \{s \in T : s$ is a limit of nodes in $\bigcup_{\alpha<\delta} \text{split}_\alpha(T)\}$.

Since there is a canonical bijection b between $2^{<\kappa}$ and $\bigcup_{\alpha<\kappa} \text{split}_\alpha(T)$ sending elements of 2^α to $\text{split}_\alpha(T)$ and recursively defined by:

- $b(\emptyset) = \text{stem}(T)$,
- $b(u^\frown i) = \text{stem}(T_{b(u)^\frown i})$ for $u \in 2^\alpha$ and $i \in 2$,
- $b(u) = \bigcup_{\beta<\alpha} b(u \restriction \beta)$ if α is a limit ordinal and $u \in 2^\alpha$,

Clearly, $|\text{split}_\alpha(T)| = |2^\alpha| = 2^{|\alpha|}$. Now, using this splitting levels we define the *fusion orderings*: given S and $T \in \mathbb{S}_\kappa$, $S \leq_\alpha T$ if and only if $S \leq T$ and $\text{split}_\alpha(T) = \text{split}_\alpha(S)$.

Definition 1.77. A *fusion sequence* of conditions $(T_\alpha : \alpha < \kappa) \subseteq \mathbb{S}_\kappa$ is sequence of conditions in $\mathbb{S}_\kappa$ such that $T_{\alpha+1} \leq_\alpha T_\alpha$ for all $\alpha < \kappa$ and for a given limit ordinal $\delta < \kappa$, $T_\delta \leq_\alpha T_\alpha$ for all $\alpha < \delta$.

Definition 1.78 (Generalized Sacks property)**.** Let $h \in \kappa^\kappa$ with $\sup_{\alpha<\kappa} h(\alpha) = \kappa$. A forcing notion $\mathbb{P}$ has the h-generalized Sacks Property if for every condition $p \in \mathbb{P}$ and every $\mathbb{P}$-name $\dot{f}$ for an element in κ^κ there are a condition $q \leq p$ and a h-slalom $F : \kappa \to [\kappa]^{<\kappa} \in \text{Loc}_h(\kappa)$ such that $q \Vdash \dot{f}(\alpha) \in F(\alpha)$ for all $\alpha < \kappa$.

Proposition 1.79. Let $h \in 2^\kappa$ the function defined by $h(\alpha) = 2^\alpha$, then $\mathbb{S}_\kappa$ has the h-generalized Sacks property. Note that, in particular this implies that κ^+ is preserved.

Proof. This proof is similar to the one for the countable case. Take $T \in \mathbb{S}_\kappa$ and $\dot{f}$ a $\mathbb{S}_\kappa$-name for an element in κ^κ, let $\mathcal{A}_\alpha$ be a maximal antichain in $\mathbb{S}_\kappa$ deciding $\dot{f}(\alpha)$. We will construct a fusion sequence of conditions $(T_\alpha : \alpha < \kappa)$ and sets $B_\alpha \subseteq \mathcal{A}_\alpha$ satisfying:

- B_α is predense below T_α.
- $|B_\alpha| \leq 2^{|\alpha|}$.

We perform the construction inductively: for the basic step notice that there exists a condition $U_0 \in \mathcal{A}_0$ compatible with T. Put T_0 to be a common extension and $B_0 = \{U_0\}$. For the successor step, suppose that T_α and B_α have been already constructed; let $u \in \text{split}_\alpha(T_\alpha)$, $i \in 2$ and consider the subtree $(T_\alpha)_{u^\frown i}$. Again, using that $\mathcal{A}_{\alpha+1}$ is a maximal antichain find a condition in it, say $U_{u^\frown i}$, compatible with $(T_\alpha)_{u^\frown i}$, take $(T_{\alpha+1})_{u^\frown i}$ to be a common extension and define $T_{\alpha+1} = \{(T_\alpha)_{u^\frown i} : u \in \text{split}_\alpha(T_\alpha)$ and $i \in 2\}$.

Thus $T_{\alpha+1} \leq_\alpha T_\alpha$ holds and we can define $B_{\alpha+1} = \{U_{u^\frown i} : u \in \text{split}_\alpha(T_\alpha) \text{ and } i \in 2\}$. Clearly this set has size at most $2^{\alpha+1}$ and by construction it is predense below $T_{\alpha+1}$.

For the limit case suppose we already have constructed $(T_\alpha : \alpha < \delta)$ where δ is a limit ordinal. Take $T_\delta = \bigcap_{\alpha<\delta} T_\alpha$ and note that for all $U \in \bigcup_{\alpha<\delta} B_\alpha$ it is possible to find $U' \in \mathcal{A}_\delta$ such that $U \parallel U'$. Choose then for each $U \in \bigcup_{\alpha<\delta} B_\alpha$ such an U' and set $B_\delta = \bigcup\{W : W$ is a common extension of both U' and $U \in \bigcup_{\alpha<\delta} B_\alpha\}$. Clearly $|B_\delta| \leq 2^\delta$.

Take the fusion S of the sequence $(T_\alpha : \alpha < \kappa)$. Then $S \leq_\alpha T_\alpha$ for all α which implies that for all α, B_α is predense below S. Hence, if we define the h-slalom $F(\alpha) = \{\beta < \alpha : \exists U \in B_\alpha : U \Vdash \dot{f}(\alpha) = \beta\}$ we have $|F(\alpha)| \leq 2^\alpha$ and $S \Vdash \dot{f}(\alpha) \in F(\alpha)$ as we wanted. $\square$

Proposition 1.80. $\mathbb{S}_\kappa$ does not have the id-generalized Sacks property. In fact, $\mathbb{S}_\kappa$ adds a function $f \in \kappa^\kappa$ such that for all $F \in \mathrm{Loc}(\kappa)$ from the ground model, $f(\alpha) \notin F(\alpha)$ for cofinally many α.

Proof. Let g be a bijection between κ and $2^{<\kappa}$. Let $\dot{s} \in 2^\kappa$ be the name for the generic κ-Sacks function. Define the name $\dot{f} \in \kappa^\kappa$ by $\dot{f}(\alpha) = g^{-1}(\dot{s} \restriction \alpha)$ for $\alpha < \kappa$. Take a slalom F from the ground model and an arbitrary condition $T \in \mathbb{S}_\kappa$. Fix a cardinal $\alpha_0 < \kappa$. We need to find $\alpha \geq \alpha_0$ and $S \leq T$ such that $S \Vdash \dot{f}(\alpha) \notin F(\alpha)$.

To this end, recursively construct a strictly increasing sequence of cardinals $(\alpha_n < \kappa : n \geq 1)$ such that $\alpha_1 > \alpha_0$ and $\mathrm{split}_{\alpha_n}(T) \subseteq 2^{\leq \alpha_{n+1}}$ for every $n \in \omega$. Put $\alpha = \sup\{\alpha_n : n \in \omega\}$. Then $\mathrm{split}_\alpha(T) = T \cap 2^\alpha$. In particular $|T \cap 2^\alpha| = |\mathrm{split}_\alpha(T)| = 2^{|\alpha|} > |\alpha|$. Hence we can find $u \in T \cap 2^\alpha$ such that $g^{-1}(u) \notin F(\alpha)$. Since $S = T_u$ forces $\dot{s} \restriction \alpha = u$, it also forces $\dot{f}(\alpha) \notin F(\alpha)$, as required. $\square$

Product and iteration

In this section we work with the following forcing notions: Let $\mathbb{Q}$ be the κ-support product of $\mathbb{S}_\kappa$ of length $\lambda > \kappa^+$, where λ is a cardinal with $\mathrm{cf}(\lambda) > \kappa$; and let also $\mathbb{P}$ to be the κ-support iteration of $\mathbb{S}_\kappa$ of length κ^{++}.

The first concern we have to take care of is the preservation of κ^+. For this matter we use the approach of both Kanamori [Kan80] and Dobrinen-Friedman in [DF10] and define a generalized fusion property which works for both forcing notions $\mathbb{Q}$ and $\mathbb{P}$.

Definition 1.81 (Similar to Definition 1.7 in [Kan80])**.**

- If $(p_\alpha : \alpha < \beta) \subseteq \mathbb{P}$ (respectively $\subseteq \mathbb{Q}$), we define a condition $p = \bigwedge_{\alpha<\beta} p_\alpha$ with $\mathrm{dom}(p) = \bigcup_{\alpha<\beta} \mathrm{dom}(p_\alpha)$ and for every $\gamma \in \mathrm{dom}(p)$, $p \restriction \gamma \Vdash_\gamma p(\gamma) = \bigcap\{p_\alpha(\gamma) : \gamma \in \mathrm{dom}(p_\alpha)\}$ (respectively $p(\gamma) = \bigcap\{p_\alpha(\gamma) : \gamma \in \mathrm{dom}(p_\alpha)\}$). Note that in the case $p \restriction \gamma \notin \mathbb{P}_\gamma$ for $\gamma \in \mathrm{dom}(p)$ or $|\mathrm{dom}(p)| > \kappa$ then p is left undefined.
- If $p, q \in \mathbb{P}$ (respectively $\in \mathbb{Q}$), $\alpha < \kappa$ and $F \subseteq \mathrm{dom}(q)$ with $|F| \leq \kappa$, we say $p \leq_{F,\alpha} q$ if and only if $p \leq q$ and for every $\beta \in F$, $p \restriction \beta \Vdash_\beta p(\beta) \leq_\alpha q(\beta)$ (respectively $p(\beta) \leq_\alpha q(\beta)$).

Claim 1.82. $\mathbb{P}$ and $\mathbb{Q}$ are κ-closed forcing notions.

Proof. Given $(p_\alpha : \alpha < \gamma) \subseteq \mathbb{P}$ (respectively $\subseteq \mathbb{Q}$) for $\gamma < \kappa$, the condition $p = \bigwedge_{\alpha<\beta} p_\alpha$ is always defined and satisfies $p \leq p_\alpha$ for all $\alpha < \gamma$. $\square$

Lemma 1.83 (Generalized fusion [Kan80])**.** Suppose $(p_\alpha : \alpha < \kappa) \subseteq \mathbb{P}$ (respectively $\subseteq \mathbb{Q}$) and $(F_\alpha \subseteq \kappa^{++} : \alpha < \kappa)$ (respectively $F_\alpha \subseteq \lambda$) have the following properties:

1. $p_\alpha \leq_{F_\alpha,\alpha} p_\alpha$ and $p_\delta = \bigwedge_{\alpha<\delta} p_\alpha$ when δ is a limit ordinal $< \kappa$.
2. $|F_\alpha| < \kappa$, $F_\alpha \subseteq F_{\alpha+1}$, $F_\delta = \bigcup_{\alpha<\delta} F_\alpha$ for limit $\delta < \kappa$ and $\bigcup_{\alpha<\kappa} F_\alpha = \bigcup_{\alpha<\kappa} \mathrm{dom}(p_\alpha)$.

Then $p = \bigwedge_{\alpha<\kappa} p_\alpha \in \mathbb{P}$ (respectively $\mathbb{Q}$). In this case we say that the sequence $(p_\alpha, F_\alpha : \alpha < \kappa)$ is a generalized fusion sequence.

And as consequence we obtain:

Corollary 1.84 (Fact 2.7 in [DF10])**.** Both forcing notions $\mathbb{P}$ and $\mathbb{Q}$ preserve cardinals $\leq \kappa^+$.

In order to decide the values of the cardinal characteristics in the diagram, it is useful as in the countable case that our forcing notions have the generalized Sacks property. We will start with the simplest case which is the product case.

Lemma 1.85. Let h be the power set function $h(\alpha) = 2^{|\alpha|}$ for all $\alpha < \kappa$. Then $\mathbb{Q}$ has the h-generalized Sacks property.

Proof. Let $\dot{f}$ be a $\mathbb{Q}$-name for a function in κ^κ and $p \in \mathbb{Q}$ a condition, we shall construct a condition $q \leq p$ and an h-slalom $G : \kappa \to [\kappa]^\kappa \in V$ such that $q \Vdash \forall\alpha < \kappa(\dot{f}(\alpha) \in G(\alpha))$.

Both the condition q and the h-slalom G will be constructed from sequences $(q_\alpha, F_\alpha : \alpha < \kappa)$ and $(B_\alpha : 0 < \alpha < \kappa) \subseteq V$ satisfying the following conditions:

1. $(q_\alpha, F_\alpha : \alpha < \kappa)$ is a generalized fusion sequence (see Lemma 1.83).
2. $q_{\alpha+1} \Vdash \dot{f}(\alpha + 1) \in B_{\alpha+1}$.
3. $|B_\alpha| \leq 2^{|\alpha|}$.

If we are able to construct such sequences, the fusion q of the sequence $(q_\alpha, F_\alpha : \alpha < \kappa)$ and the slalom defined as $G(\alpha) = B_\alpha$ are the witnesses we are looking for to guarantee the h-generalized Sacks property. The rest of the proof deals with how to carry out the construction of these sequences by induction on $\alpha < \kappa$:

- **Basic step**: Start with $q_0 = p$ and $F_0 = \emptyset$.
- **Successor step**: Suppose we already have defined q_α, F_α and B_α satisfying (1.-3.) and consider the set $\Lambda = \{\bar{\sigma} = (\sigma_0, \sigma_1, \ldots, \sigma_i, \ldots)_{i\in F_\alpha} : \sigma_i \in \text{split}_\alpha(q_\alpha(i))\}$ which has size $< \kappa$ because $|\text{split}_\alpha(q_\alpha(i))| \leq 2^{|\alpha|}$ and $|F_\alpha| < \kappa$. Also take $\{\bar{\sigma}_l : l < \gamma\}$ for some $\gamma < \kappa$ to be an enumeration of Λ. We construct a sequence $\{(r_\alpha)^h_{\bar{\sigma}_l} : h : F_\alpha \to 2$ and $l < \gamma\}$ of conditions in $\mathbb{Q}$ in the following way: Consider the sequence $\bar{\sigma_0}$ and $h : F_i \to 2$ and define the condition:

$$(q_\alpha)^h_{\bar{\sigma}_0}(i) = \begin{cases} (q_\alpha(i))_{\sigma_i^\frown h(i)} & \text{if } i \in F_\alpha \\ q_\alpha(i) & \text{otherwise} \end{cases}$$

  Below this condition there exists $(r_\alpha)^h_{\bar{\sigma}_0} \leq (q_\alpha)^h_{\bar{\sigma}_0}$ such that $(r_\alpha)^h_{\bar{\sigma}_0} \Vdash \dot{f}(\alpha) = \gamma^h_{\bar{\sigma}_0}$. If we have constructed $\{(r_\alpha)^h_{\bar{\sigma}_j} : h : F_\alpha \to 2$ and $j < l\}$ define the following condition:

$$(q_\alpha)^h_{\bar{\sigma}_l}(i) = \begin{cases} (q_\alpha(i))_{\sigma_i^\frown h(i)} & \text{if } i \in F_\alpha \text{ and } \bar{\sigma}_l(i) \neq \bar{\sigma}_j(i) \text{ for all } j < l \\ (r_\alpha)^h_{\bar{\sigma}_j}(i) & \text{if } i \in F_\alpha \text{ and } j = \sup\{k < l : \bar{\sigma}_l(i) = \bar{\sigma}_k(i)\} \\ q_\alpha(i) & \text{otherwise} \end{cases}$$

  Then take as in the base case $(r_\alpha)^h_{\bar{\sigma}_l} \leq (q_\alpha)^h_{\bar{\sigma}_l}$ such that $(r_\alpha)^h_{\bar{\sigma}_l} \Vdash \dot{f}(\alpha) = \gamma^h_{\bar{\sigma}_l}$.

Put $q_{\alpha+1} = \bigcup\{(r_\alpha)^h_{\bar\sigma_l} : h \in 2^{F_i}$ and $l < \gamma\}$, our construction ensures that this is indeed a condition in $\mathbb{Q}$. Finally put $F_{\alpha+1} = F_\alpha \cup \{$ the first α-many elements in $\mathrm{supp}(q_{\alpha+1} \setminus F_\alpha)\}$ and $B_\alpha = \{\gamma^h_{\bar\sigma_l} : h \in 2^{F_i}$ and $l < \gamma\}$. Clearly the conditions (1)-(3) are satisfied.

- **Limit Step**: Let $\delta < \kappa$ be a limit ordinal, the definitions of q_δ and F_δ are already given if we want to achieve that $(q_\alpha, F_\alpha : \alpha < \kappa)$ is a generalized fusion sequence. To define B_δ we repeat the same construction that we used for the limit step for each $i \in F_\delta = \bigcup_{\alpha<\delta} F_\alpha$ and every $\bar\sigma \in \prod_{i\in F_\delta} \mathrm{split}_\delta(q_\delta(i))$.

$\square$

A bit more technical but in essence the same argument as above, allows us to conclude also that:

Proposition 1.86. Let h be the power set function $h(\alpha) = 2^{|\alpha|}$ for all $\alpha < \kappa$, then $\mathbb{P}$ has the h-generalized Sacks Property.

Proof. The first part of the proof is basically the same. Given $\dot f$ a $\mathbb{P}$-name for a function in κ^κ and $p \in \mathbb{P}$ we will construct sequences $(q_\alpha, F_\alpha : \alpha < \kappa)$ and $(B_\alpha : 0 < \alpha < \kappa) \subseteq V$ satisfying the conditions (1)-(3) above. We focus on the process to carry out the inductive construction.

- **Basic step**: Start with $q_0 = p$ and $F_0 = \emptyset$.
- **Successor step**: Suppose we already have defined q_α, F_α and B_α satisfying (1)-(3). For each $i \in F_\alpha$, $q_\alpha(i)$ is a $(\mathbb{P} \restriction i)$-name for a κ-Sacks tree. As in the case of products we want to look at the α-th splitting level for each coordinate $q_\alpha(i)$ for $i \in F_\alpha$, the problem is that now we are dealing with names. In consequence, we use the method of canonical names introduced by Dobrinen-Friedman in [DF10]. For each $i \in F_\alpha$ and $\xi \in 2^\alpha$ let $\dot s_\alpha(i,\xi)$ be a $(\mathbb{P} \restriction i)$-name for the ξ-th element in $\mathrm{split}_\alpha(q_\alpha(i))$ under the natural bijection between $\mathrm{split}_\alpha(q_\alpha(i))$ and 2^α. Also let $\tilde r^m_\alpha(i,\xi)$ be any $(\mathbb{P} \restriction i)$-name for a Sacks tree contained in $q_\alpha(i)$ with stem containing $\dot s_\alpha(i,\xi)^\frown m$, for $m \in \{0,1\}$. Then $r_{\alpha+1} = \bigcup\{\tilde r^m_\alpha(i,\xi) : \xi \in 2^\alpha$ and $n \in \{0,1\}\}$ is the canonical $(\mathbb{P} \restriction i)$-name for a κ-Sacks tree such that it is forced to be equal to $q_\alpha(i) \restriction \dot s_\alpha(i,\xi)$.

  Now consider $\Lambda = \{\bar\sigma = (\zeta_0, \zeta_1, \ldots, \zeta_i, \ldots)_{i\in F_\alpha} : \zeta_i \leq 2^\alpha\}$ and as in the case of the products let $\{\bar\sigma_l : l < \gamma\}$ for some $\gamma < \kappa$ be an enumeration of it. (It is clear that this set has size $<\kappa$).

  We construct a sequence $\{(\dot r_\alpha)^h_{\bar\sigma_l} : h : F_\alpha \to 2$ and $l < \gamma\}$ of conditions in $\mathbb{P}$ in the following way: Consider the sequence $\bar\sigma_0$ and $h : F_i \to 2$ and define the condition,

$$(q_\alpha)^h_{\bar\sigma_0}(i) = \begin{cases} \tilde r^{h(i)}_\alpha(i,\xi) & \text{if } i \in F_\alpha. \\ q_\alpha(i) & \text{otherwise .} \end{cases}$$

  Below this condition there exists $(r_\alpha)^h_{\bar\sigma_0} \leq (q_\alpha)^h_{\bar\sigma_0}$ such that $(r_\alpha)^h_{\bar\sigma_0} \Vdash \dot f(\alpha) = \gamma^h_{\bar\sigma_0}$.

For the induction step we argue as in the case for the products just taking care of choosing the conditions $(q_\alpha)^h_{\bar{\sigma}_l}(i)$ using the canonical names. At the end, we define $q_{\alpha+1}$, $F_{\alpha+1}$ and $B_{\alpha+1}$ as in the product case.

- Limit Step: Again we do the same as in the product case but we now use both the canonical bijection between $\mathrm{split}_\delta(q_\delta(i))$ and 2^δ to enumerate all sequences $\Lambda = \{\bar{\sigma} = (\zeta_0, \zeta_1, \ldots, \zeta_i, \ldots)_{i \in F_\delta} : \zeta_i \leq 2^\alpha\}$ and the canonical names $\tilde{r}_\delta^{h(i)}(i, \xi)$ for $h \in 2^{F_\alpha}$ and $\xi \in 2^\alpha$.

$\square$

The effect of both forcings $\mathbb{Q}$ and $\mathbb{P}$ in the generalized Cichoń's diagram is similar to the countable case. Moreover, if we consider the extended diagram we obtain the following:

Theorem 1.87. *Assume* GCH, *forcing with* $\mathbb{Q}$ *yields to a generic extension in which* $\mathfrak{d}_{\mathrm{id}}(\in^*_p)(\kappa) = \mathfrak{d}_h(\in^*)(\kappa) = \kappa^+$ *and* $\mathfrak{d}_{\mathrm{id}}(\in^*) = 2^\kappa = \lambda$.

Proof. $\mathfrak{d}_h(\in^*)(\kappa) = \kappa^+$ follows immediately from Lemma 1.85. By earlier results, this also implies that all other cardinals we have considered are equal to κ^+. Furthermore, $2^\kappa = \lambda$ is an easy consequence of GCH and $\mathrm{cf}(\lambda) > \kappa$, using Lemma 1.85 in the $\mathrm{cf}(\lambda) = \kappa^+$ case. Hence we are left with showing $\mathfrak{d}_{\mathrm{id}}(\in^*) \geq \lambda$. This follows almost, but – since we are dealing with a product and not an iteration – not quite, from Proposition 1.80.

Assume $\mu < \lambda$, and let $\{\dot{F}_\gamma : \gamma < \mu\}$ be $\mathbb{Q}$-names for slaloms in $\mathrm{Loc}(\kappa)$. By the κ^{++}-cc, we may find sets $A_\gamma \subseteq \lambda$, $\gamma < \mu$, of size at most κ^+ such that $\dot{F}_\gamma$ is added by the sub-forcing $\mathbb{Q}_{A_\gamma}$. This subforcing of $\mathbb{Q}$ corresponds to the set of all functions $p : A_\gamma \to \mathbb{S}_\kappa$ with $\mathrm{supp}(p)$ of size $<\kappa$.

Fix $\beta \in \lambda \setminus \bigcup_{\gamma<\mu} A_\gamma$. Let $\dot{s}_\beta \in 2^\kappa$ be the name for the generic κ-Sacks function added in coordinate β. Define the name $\dot{f} \in \kappa^\kappa$ by $\dot{f}(\alpha) = g^{-1}(\dot{s}_\beta \restriction \alpha)$ for $\alpha < \kappa$ where g is again a bijection between κ and $2^{<\kappa}$, like in Proposition 1.80. We prove now that $\dot{f}$ is forced not to be localized by any $\dot{F}_\gamma$. This is clearly sufficient.

Fix $\gamma < \mu$. Also fix $p \in \mathbb{Q}$ and a cardinal $\alpha_0 < \kappa$. We need to find $\alpha \geq \alpha_0$ and $q \leq p$ such that $q \Vdash \dot{f}(\alpha) \notin \dot{F}_\gamma(\alpha)$. Let $p^0 = p \restriction A_\gamma \in \mathbb{Q}_{A_\gamma}$. Construct a decreasing chain $(p^0_\alpha : \alpha < \kappa)$ of conditions in $\mathbb{Q}_{A_\gamma}$ with $p^0_0 = p^0$ and a slalom $F \in \mathrm{Loc}(\kappa)$ such that $p^0_\alpha \Vdash \dot{F}_\gamma \restriction \alpha = F \restriction \alpha$.

Next, as in the proof of Proposition 1.80, recursively construct a strictly increasing sequence of cardinals $(\alpha_n < \kappa : n \geq 1)$ such that $\alpha_1 > \alpha_0$ and $\mathrm{split}_{\alpha_n}(p(\beta)) \subseteq 2^{\leq \alpha_{n+1}}$ for every $n \in \omega$. Put $\alpha = \sup\{\alpha_n : n \in \omega\}$. Again we see that $|p(\beta) \cap 2^\alpha| = |\,\mathrm{split}_\alpha(p(\beta))| = 2^{|\alpha|} > \alpha$. Hence we can find $u \in p(\beta) \cap 2^\alpha$ such that $g^{-1}(u) \notin F(\alpha)$. Now define a condition q by:

- $q(\beta) = (p(\beta))_u$,
- $q \restriction A_\gamma = p^0_{\alpha+1}$,
- $q(\delta) = p(\delta)$ for $\delta \notin A_\gamma \cup \{\beta\}$.

Clearly q forces $\dot{s}_\beta \restriction \alpha = u$ and, thus $\dot{f}(\alpha) = g^{-1}(u) \notin F(\alpha) = \dot{F}_\gamma(\alpha)$, as required. $\square$

The only difference between the forcings $\mathbb{Q}$ and $\mathbb{P}$ is the value of 2^κ. In the extension from the theorem above $2^\kappa = \lambda$ whereas that when evaluating the value of 2^κ in the extension obtained by $\mathbb{P}$ its value is κ^{++}. Summarizing, we have the following diagram:

$$
\begin{tikzcd}
 & & & \operatorname{non}\mathcal{M}(\kappa) \arrow[r] & \operatorname{cof}\mathcal{M}(\kappa) \arrow[r] & \mathfrak{d}(\in^*_p)(\kappa) \arrow[r] & \mathfrak{d}(\in^*)(\kappa) \arrow[r] & 2^\kappa \\
 & & & \mathfrak{b}(\kappa) \arrow[u] \arrow[r] & \mathfrak{d}(\kappa) \arrow[u] & & & \\
\kappa^+ \arrow[r] & \mathfrak{b}(\in^*)(\kappa) \arrow[r] & \mathfrak{b}(\in^*_p)(\kappa) \arrow[r] & \operatorname{add}\mathcal{M}(\kappa) \arrow[u] \arrow[r] & \operatorname{cov}\mathcal{M}(\kappa) \arrow[u] & & &
\end{tikzcd}
$$

Figure 1.9.: Extended Diagram and the effect after forcing with $\mathbb{P}$

1.3.2. κ-Miller forcing

Let κ be a measurable cardinal and $\mathcal{F}$ be a normal κ-complete filter on κ. We follow a similar approach to [FHZ13] to define the generalization of Miller forcing $\mathbb{MI}^\kappa_\mathcal{F}$ for uncountable κ.

Definition 1.88 (Generalized Miller forcing)**.** Conditions in $\mathbb{MI}^\kappa_\mathcal{F}$ are subtrees T of the set of the increasing sequences in $\kappa^{<\kappa}$ satisfying the following conditions:

1. If $s \in T$ and $t \subseteq s$, then $t \in T$.
2. For every $\alpha < \kappa$ limit ordinal and $s \in \kappa^\alpha$. If $s \restriction \beta \in T$ for all $\beta < \alpha$, then $s \in T$.
3. For every $s \in T$, there is an $\mathcal{F}$-splitting node $t \supseteq s$ (meaning a node with filter-many immediate successors). Moreover, if t_0, t_1 are splitting nodes such that $t_0 \subseteq t_1$, then $F_0 \subseteq F_1$, where $F_i = \{\alpha < \kappa : s^\frown\alpha \in \operatorname{succ}_T(t_i)\} \in \mathcal{F}$ and $i = 0, 1$.
4. The limit of $\mathcal{F}$-splitting nodes is $\mathcal{F}$-splitting.

The order is inclusion, i.e. $S \leq T$ if and only if $S \subseteq T$.

Since the ultrafilter $\mathcal{F}$ is κ-complete it is easy to see that $\mathbb{MI}^\kappa_\mathcal{F}$ is $<\kappa$-closed. As for κ-Sacks forcing, if $T \in \mathbb{MI}^\kappa_\mathcal{F}$ and $u \in T$ we put $T_u = \{t \in T : t \subseteq u \text{ or } u \subseteq t\}$. For $\mathbb{MI}^\kappa_\mathcal{F}$, $\operatorname{stem}(T)$ is analogously the unique splitting node that is comparable with all elements in T.

Also let $\operatorname{split}(T)$ be the set of all $u \in T$ which split in T. Given $u \in \kappa^{<\kappa}$ let $|u|$ denote the *length* of u, i.e. the unique α such that $u \in \kappa^\alpha$.

Definition 1.89 (The α-th splitting level of T, see [FZ10])**.** Let $\mathrm{split}_\alpha(T)$ be the set of all $u \in \mathrm{split}(T)$ such that:

- $\{v \in \mathrm{split}(T) : v \subsetneq u\}$ has order type at most α and
- for all $v \subsetneq u$ in $\mathrm{split}(T)$, $u(|v|)) \cap \mathrm{succ}_T(v)$ has order type at most α.

Thus $\mathrm{split}_0(T) = \{\mathrm{stem}(T)\}$ and we note for later use that $|\mathrm{split}_\alpha(T)| \leq |(\alpha + 1)^{\alpha+1}|$. Also $\mathrm{split}_\alpha(T) \subset \mathrm{split}_\beta(T)$ for $\alpha < \beta$ and $\mathrm{split}(T)$ is the increasing union of the $\mathrm{split}_\alpha(T)$'s. Using again the definition of the splitting levels $\mathrm{split}_\alpha(T)$, define the *fusion orderings* $\leq_\alpha$ for $\alpha < \kappa$ by $S \leq_\alpha T$ if and only if $S \leq T$ and $\mathrm{split}_\alpha(T) = \mathrm{split}_\alpha(S)$.

Definition 1.90 (Fusion sequence)**.** A sequence of conditions $(T_\alpha : \alpha < \kappa) \subseteq \mathbb{MI}^\kappa_{\mathcal{F}}$ is a *fusion sequence* if:

- $T_{\alpha+1} \leq_\alpha T_\alpha$ for every $\alpha < \kappa$,
- $T_\delta = \bigcap_{\alpha<\delta} T_\alpha$ for limit $\delta < \kappa$.

Claim 1.91. Given a fusion sequence $(T_i : i < \kappa)$, the fusion of the sequence $T = \bigcap_{i<\kappa} T_i$ is a condition in $\mathbb{MI}^\kappa_{\mathcal{F}}$.

Proof. Conditions (1.) and (2.) are immediate. For (3.) note that given a condition $T \in \mathbb{MI}^\kappa_{\mathcal{U}}$ and $s \in T$, if we consider j to be the unique ordinal such that $s \in j^{\leq j}$; in the tree T_j there exists $t \supseteq s$ such that $t \in T_j$ and $t \in \mathrm{split}(T_j)$. Let $i < \kappa$ be such $t \in i^{\leq i}$, then $t \in \mathrm{split}_i(T_i)$ and since $T \leq_i T_i$ we obtain $\mathrm{split}_i(T) = \mathrm{split}_i(T_i)$. This implies that given a node in the tree T we can find a node $t \in T$ such that t is a splitting node.

To conclude that the fusion is a condition, we have to show that if a node $s \in \mathrm{split}(T_i)$ for all $i < \kappa$ then $s \in \mathrm{split}(T)$. To this end take $F_i \in \mathcal{U}$ to be the set $F_i = \{\alpha < \kappa : s^\frown\alpha \in \mathrm{succ}_{T_i}(s)\}$, then $\triangle_{i<\kappa} F_i \subseteq \{\alpha < \kappa : s^\frown\alpha \in \mathrm{succ}_T(s)\}$ because since $T_i \leq_i T$ for all $i < \kappa$ we have $F_i \cap i \subseteq F_{i+1}$.

Condition (4.) follows from the reasoning before, if $s = \sup_{i<j} s_i$ where $s_i \in \mathrm{split}(T)$, then $s_i \in \mathrm{split}(T_l)$ for all $l < \kappa$ and then s is a splitting node in all the trees T_i, so it belongs to T. □

Note: From now on, we will be mostly interested in the ultrafilter version of Miller forcing. For cardinal preservation results, it is indeed enough to consider a normal filter on κ (See [FHZ13]). However κ-Miller forcing without an ultrafilter is extremely different from the countable case, namely, it adds Cohen subsets of κ.

Theorem 1.92. *κ-Miller forcing with the club filter $\mathcal{C}$ adds a Cohen subset of κ.*

Proof. Let $(S_\alpha : \alpha < \kappa)$ be a partition of κ in stationary sets. Let $f(\beta)$ be the unique α such that $\beta \in S_\alpha$ and consider $m_{\mathcal{C}}$ to be the generic function added by $\mathbb{MI}^\kappa_{\mathcal{C}}$.

In $V^{\mathbb{MI}^\kappa_{\mathcal{C}}}$ define the function $g^* : \kappa \to 2^{<\kappa}$ as the composition $g^* = \varphi \circ f \circ m_{\mathcal{C}}$ where φ is a bijection between κ and $2^{<\kappa}$. We claim that the function $g : \kappa \to 2$ that concatenates the values of g^*, $g(\alpha) = g^*(0)^\frown g^*(1)^\frown \dots g^*(\alpha)$ is κ-Cohen generic over the ground model V. For this purpose take $T \in \mathbb{MI}^{\kappa}_{\mathcal{C}}$ and $\mathcal{D} \subseteq \mathbb{C}_{\kappa}$ open dense, it is then enough to show that we can find $u \in \mathcal{D}$ and $S \leq T$ with the property $S \Vdash u \subseteq \dot{g}$.

Put $\sigma = \text{stem}(T)$ and consider the composition $v^* = \varphi \circ f \circ \sigma \in (2^{<\kappa})^{<\kappa}$ and let v be its concatenation, then $v \in \mathbb{C}_{\kappa}$ and so there exists $u \in D$ such that $u \supseteq v$. Note that $T \Vdash m_{\mathcal{C}} \restriction_{|\sigma|} = \sigma$, so our desired tree $S \leq T$ will have stem $\tau \supseteq \sigma$ satisfying $S \Vdash \forall \alpha \in \text{dom}(v^*)(v^*(\alpha) = \text{the concatenation of the values } \varphi(f(\tau(\beta))) \text{ for } \beta < \alpha)$.

Look at the value $\varphi^{-1}(u(|\sigma|)) = \beta_{\sigma}$ and define $\tau(|\sigma|)$ to be any element of $S_{\beta_{\sigma}} \cap C_{\sigma} = \{\delta : \sigma^\frown\delta \in \text{succ}_T(\sigma)\}$, this is possible because C_{σ} is a club subset of κ. Now look at the next splitting node σ_1 in T above $\sigma^\frown\tau(|\sigma|) = \sigma^*$ and extend u to a condition $u_1 \in D$ such that the concatenation of the elements in $\varphi(f(\sigma_1))$ belongs to $\text{ran}(u_1)$ and define $\tau \restriction (|\sigma_1| \backslash |\sigma^*|) = \sigma_1$. Take $\gamma \in \text{dom}(u) \setminus \gamma$ where $\gamma = \min\{j \in \text{dom}(u) \setminus \text{dom}(v)\}$ and repeat the same process with the splitting node σ_1 instead of σ.

Finally, note that this process can be iterated and it finishes in $<\kappa$-many steps, as many as $|\text{dom}(u) \setminus \text{dom}(v)|$, and gives us the sequence τ that we want. The tree S is defined as T_{τ} clearly forces $u \subseteq \dot{g}$. $\square$

From now on we work with the ultrafilter version of κ-Miller forcing.

Proposition 1.93. $\mathbb{MI}^{\kappa}_{\mathcal{U}}$ generically adds an unbounded function in κ^{κ}.

Proof. Given a function $g \in \kappa^{\kappa}$ the set $D_g = \{T \in \mathbb{MI}^{\kappa}_{\mathcal{U}} : \forall s \in \text{split}(T) \forall \alpha < \kappa \text{ such that } s^\frown\alpha \in \text{succ}_T(s)(g(|s|) < \alpha)\}$ is dense in $\mathbb{MI}^{\kappa}_{\mathcal{U}}$. Given $T \in \mathbb{MI}^{\kappa}_{\mathcal{U}}$, define a tree $S \leq T$ as the fusion of the sequence $(S_i : i < \kappa)$ constructed as follows:

Start with $T_0 = T$; for the successor step take T_i and consider $s \in \text{split}_i(T)$. The set $X_s = \text{succ}_T(s) \cap \{\alpha < \kappa : \alpha > g(|s|)\}$ belongs to $\mathcal{U}$ and we have that the tree $T_{i+1} = \bigcup_{s \in \text{split}_i(T)} \bigcup_{\alpha \in X_s} T_{\sigma^\frown\alpha} \leq T_i$. In the limit steps take intersections.

At the end, the fusion of the sequence S is a κ-Miller tree that belongs to D_g and is stronger than T. This immediately allow us to conclude that given G a $\mathbb{MI}^{\kappa}_{\mathcal{U}}$-generic filter, the generic function added by κ-Miller forcing $m_i = \bigcap\{[T] : T \in G\}$ is unbounded over the ground model ones. $\square$

Proposition 1.94 (Pure decision property)**.** Let φ be a sentence of the forcing language and assume $T \in \mathbb{MI}^{\kappa}_{\mathcal{U}}$. Then there is $S \leq T$ with the same stem such that S decides φ, meaning $S \Vdash \varphi$ or $S \Vdash \neg\varphi$.

Proof. Let T and φ as in the proposition and put $\sigma = \text{stem}(T)$, then there is a set $X \in \mathcal{U}$ such that $\sigma^\frown i \in T$ for all $i \in X$. Given $i \in X$ find $S_i \leq T_{\sigma^\frown i}$ such that S_i decides φ. Since $\mathcal{U}$ is ultra one of the sets $X_0 = \{i \in X : S_i \Vdash \varphi\}$ or $X_1 = \{i \in X : S_i \Vdash \neg\varphi\}$ must belong to $\mathcal{U}$. Hence, it suffices to define the condition S that we want as follows: $S = \bigcup_{i \in X^*} S_i$, where X^* is either X_0 or X_1 depending which one of them belongs to $\mathcal{U}$. Clearly, $S \leq T$ and S decides φ, specifically if $X_0 \in \mathcal{U}, S \Vdash \varphi$, otherwise $S \Vdash \neg\varphi$. $\square$

Definition 1.95 (Generalized h-Laver property)**.** Let $h \in \kappa^\kappa$ with $\sup_{\alpha<\kappa} h(\alpha) = \kappa$. A forcing notion $\mathbb{P}$ has the *h-generalized Laver property* if for every condition $p \in \mathbb{P}$, every $g \in V \cap \kappa^\kappa$ and every $\mathbb{P}$-name $\dot{f}$ for an element in κ^κ such that $\Vdash_{\mathbb{P}} \forall \alpha < \kappa(\dot{f}(\alpha) \leq g(\alpha))$ there are a condition $q \leq p$ and an h-slalom $F : \kappa \to [\kappa]^{<\kappa}$ such that both $|F(\alpha)| \leq 2^{|\alpha|}$ and $q \Vdash \dot{f}(\alpha) \in F(\alpha)$ for all $\alpha < \kappa$.

The generalized h-Laver property is closely related to the generalized Sacks property: Say that a forcing notion $\mathbb{P}$ is *κ^κ-bounding* if for all $\mathbb{P}$-names $\dot{f} \in \kappa^\kappa$ and all $p \in \mathbb{P}$, there are $g \in \kappa^\kappa$ and $q \leq p$ such that $q \Vdash \dot{f}(\alpha) < g(\alpha)$ for all $\alpha < \kappa$. Then $\mathbb{P}$ has the generalized h-Sacks property if and only if $\mathbb{P}$ is κ^κ-bounding and has the generalized h-Laver property.

Analogous to the countable case we have the following property:

Proposition 1.96. Let $h \in \kappa^\kappa$ with $\sup_{\alpha<\kappa} h(\alpha) = \kappa$. If $\mathbb{P}$ is κ-closed and has the h-generalized Laver property, then $\mathbb{P}$ does not add κ-Cohen functions.

Proof. For convenience we will work in the space 2^κ instead of κ^κ. Take $\dot{f}$ to be a $\mathbb{P}$-name for a function in 2^κ, and consider a partition of κ in κ-many intervals $(I_\alpha : \alpha < \kappa)$ such that $|I_\alpha| = |h(\alpha)|$. Now define the function $\dot{g} \in 2^\kappa$, $\dot{g}(\alpha) = \dot{f} \restriction I_\alpha$, note that $\dot{g}$ does not belong to the ground model V, however it is possible to find a function $G \in V$ that bounds $\dot{g}$.

Since the possible values of $\dot{g}(\alpha)$ are bounded by the number of functions from the interval I_α to 2 which is $2^{|I_\alpha|} = 2^{|h(\alpha)|} < \kappa$ (κ is strongly inaccessible), we can define G as $G(\alpha) = 2^{I_\alpha}$.

Thus we have that $\Vdash_{\mathbb{P}} \dot{g} \leq G$ (everywhere). Then using the Generalized Laver Property, given a condition $p \in \mathbb{P}$ we can find an h-slalom $F : \kappa \to [\kappa]^{<\kappa}$ such that $|F(\alpha)| \leq 2^{|\alpha|}$ and $q \Vdash \dot{g}(\alpha) \in F(\alpha)$ for all $\alpha < \kappa$. Thus, if we define the set $A = \{h \in 2^\kappa : (\exists \beta)(\forall \alpha > \beta)(h \restriction I_\alpha \in F(\alpha))\} = \bigcup_{\alpha<\kappa} \bigcap_{\beta>\alpha} \{h \restriction I_\alpha \in F(\alpha)\}$ it is enough to notice that every set of the form $\bigcap_{\beta>\alpha} \{h \restriction I_\alpha \in F(\alpha)\}$ for $h \in 2^\kappa$ and β fixed is nowhere dense and also that $\dot{f} \in A$, this implies that $\dot{f}$ is not κ-Cohen. $\square$

Proposition 1.97. Let $h \in \kappa^\kappa$ be the power set function $h(\alpha) = 2^\alpha$. If $\mathcal{U}$ is a normal measure on κ then κ-Miller forcing $\mathbb{MI}^\kappa_{\mathcal{U}}$ has the h-generalized Laver property.

Proof. Let $\dot{f}$ be a $\mathbb{MI}^\kappa_{\mathcal{U}}$-name for an element in κ^κ, $g \in \kappa^\kappa \cap V$ satisfying $\Vdash_{\mathbb{MI}^\kappa_{\mathcal{U}}} \dot{f} \leq^* g$ and $T \in \mathbb{MI}^\kappa_{\mathcal{U}}$.

Inductively, we construct a fusion sequence $(S_i : i < \kappa)$ and sets $B_i \in V$ such that:

1. $S_{i+1} \leq_i S_i$ for all $i < \kappa$.
2. $S_{i+1} \Vdash \dot{f}(i) \in B_i$.
3. $|B_i| \leq 2^{|i|}$.

Start with $S_0 = T$. For the successor case, suppose we have already S_i and take $t \in \mathrm{split}_i(S_i) = \mathrm{split}(S_i) \cap i^{<i}$; recall that the set $\mathrm{next}_i(t) = \{j < \kappa : t^\frown j \in \mathrm{succ}_{S_i}(t)\} \in \mathcal{U}$. Hence for all $j \in \mathrm{next}_i(t)$ such that $j > i+1$ we can find a condition $U_{t^\frown j} \leq (S_i)_{t^\frown j}$ such that $U_{t^\frown j} \Vdash \dot{f}(i+1) = \gamma_i^j$.

Note that $A_i = \{\gamma_i^j : j \in \mathrm{next}_i(t) \text{ and } j > i+1\}$ has size bounded by $|g(i)|$. So, there exists a set $D_i \subseteq \mathrm{next}_i(t) \cap \{j : j > i+1\}$ in $\mathcal{U}$ such that for all $j \in D_i$, $U_{t^\frown j} \Vdash \dot{f}(i) = \gamma_i$ for a single $\gamma_i^t < g(i)$. Thus, we can find a condition $W_i^t = \bigcup_{j \in D_i} U_{t^\frown j} \cup \bigcup_{j \leq i} V_{t^\frown j}$ which forces a small set of values for $\dot{f}(i+1)$. Here $V_{t^\frown j}$ is a condition below $(S_i)_{t^\frown j}$ forcing $\dot{f}(\alpha+1)$ to be an ordinal $\beta_j^t < \kappa$.

At the end it is enough to define $S_{i+1} = \bigcup_{t \in \mathrm{split}_i^*(S_i)} W_i^t$. This is a κ-Miller tree with $S_{i+1} \leq_i^* S_i$: define also $B_i = \{\gamma_i^t : t \in \mathrm{split}_i(S_i)\} \cup \{\beta_j^t : j \leq i \text{ and } t \in \mathrm{split}_i(S_i)\}$. This set is clearly bounded by $2^{|i|}$.

For i a limit ordinal define $S_i = \bigcap_{j<i} S_j$ and take W to be a condition in $\mathcal{A}_i$ compatible with some $U \leq S_i$ and set B_i to be the set of values $\gamma < \kappa$ such that some W as before forces $\dot{f}(i) = \gamma$. Clearly $|B_i| \leq |\bigcup_{j<i} B_j| \leq 2^i$.

Finally, take the fusion S of the sequence $(S_i : i < \kappa)$. Then $S \leq_i S_i$ for all i which implies that for all $i < \kappa$, $S \Vdash \dot{f}(i) \in B_i$. Hence, if we define the h-slalom $G(i) = \{\gamma : \exists U \leq S_i \text{ and } U \Vdash \dot{f}(i) = \gamma\}$. Then $|G(i)| \leq 2^{|i|}$ and $S \Vdash \dot{f}(i) \in G(i)$, so we have the h-generalized Laver property. □

Our reason for investigating the h-generalized Laver property for forcings of type $\mathbb{MI}_{\mathcal{U}}^{\kappa}$ was the hope to obtain an alternative proof for the consistency of $\mathrm{cov}\,\mathcal{M}(\kappa) < \mathfrak{d}(\kappa)$, originally obtained by Shelah [She13] (see [BJ95, 7.3.E] for $\mathrm{cov}(\mathcal{M}) < \mathfrak{d}$ in the Miller model). For this, however, one would need also the preservation of this property along iterations or products of support of size κ.

The product

As in the previous cases, we want to consider ways of adding many κ-Miller functions. Unlike the countable case, where the product with countable support of classic Miller forcing collapses cardinals, in the uncountable case, it preserves cardinals below κ^+. Therefore, we consider the product $\mathbb{Q}$ of κ-Miller forcing $\mathbb{MI}_{\mathcal{U}}^{\kappa}$ where $\mathcal{U}$ is a normal ultrafilter in the ground model, with κ-support and length $\lambda > \kappa^+$ (here λ is a cardinal with $\mathrm{cf}(\lambda > \kappa)$).

In the countable case we have that although the product of two copies of Miller forcing $\mathbb{MI}^2$ does not add Cohen reals (Spinas [Spi01]), the product of three copies of Miller forcing $\mathbb{MI}^3$ adds a Cohen real (Veličković and Woodin [VW98]). For strongly inaccessible κ, two κ-Miller functions are sufficient to get a κ-Cohen function. Roughly speaking, the reason for this is that given trees T and $S \in \mathbb{MI}_{\mathcal{F}}^{\kappa}$, the set of places where both T and S split contains a club.

Proposition 1.98. Let $\mathcal{F}$ be a κ-complete filter, then the product $\mathbb{Q}^* = \mathbb{MI}^{\kappa}_{\mathcal{F}} \times \mathbb{MI}^{\kappa}_{\mathcal{F}}$ adds a κ-Cohen function.

Proof. Let m_0 and m_1 be the generically added κ-Miller functions. We say that $\alpha < \kappa$ is an oscillation point of m_0 and m_1 if and only if $\exists \gamma < \alpha \forall \gamma \leq \beta < \alpha$ such that either $(m_0(\beta) < m_1(\beta)$ and $m_0(\alpha) > m_1(\alpha))$ or $(m_1(\beta) < m_0(\beta)$ and $m_1(\alpha) > m_0(\alpha))$. We also denote A to be the set of oscillation points of m_0 and m_1 and C to be the set of limit points of A. Then we have the following:

Claim 1.99. *C is a club in κ.*

Proof. We show that the set:

$$\mathcal{D} = \{(S,T) \in \mathbb{Q} : \exists \alpha \in \mathrm{dom}(\mathrm{stem}(S)) = \mathrm{dom}(\mathrm{stem}(T)) \mathrm{and} \alpha \in C\} \text{ is dense in } \mathbb{Q}$$

Take (S,T) to be an arbitrary condition in $\mathbb{Q}$ and put $\sigma = \mathrm{stem}(S)$ and $\tau = \mathrm{stem}(T)$. First note that it is dense to assume that $\mathrm{dom}(\sigma) = \mathrm{dom}(\tau)$. If $\mathrm{dom}(\sigma) \subsetneq \mathrm{dom}(\tau)$ it is possible to construct a sequence of ordinals $(\alpha_n : n \in \omega) \subseteq \kappa$ and a sequence $(s_n : n \in \omega)$ of splitting nodes such that $(s_{2n} : n \in \omega) \in \mathrm{split}(S)$ and $(s_{2n+1} : n \in \omega) \in \mathrm{split}(T)$ as follows: $\alpha_0 = \mathrm{dom}(\sigma)$, $\alpha_1 = \mathrm{dom}(\tau)$, $s_0 = \sigma$ and $s_1 = \tau$. If α_{2n}, s_{2n} and α_{2n+1}, s_{2n+1} have been already constructed, take $s_{2n+2} \in \mathrm{split}(S)$ extending s_{2n} such that $\sup(\mathrm{dom}(s_{2n+2})) > \alpha_{2n+1}$ and put $\alpha_{2n+2} = \sup(\mathrm{dom}(s_{2n+2}))$. In a similar fashion choose $s_{2n+3} \in \mathrm{split}(T)$ extending s_{2n+1} such that $\sup(\mathrm{dom}(s_{2n+3})) > \alpha_{2n+2}$ and $\alpha_{2n+3} = \sup(\mathrm{dom}(s_{2n+3}))$.

At the end put $\alpha = \sup_{n\in\omega} \alpha_{2n} = \sup_{n\in\omega} \alpha_{2n+1}$, $s = \bigcup_{n\in\omega} s_{2n} \in \mathrm{split}(S)$ and $t = \bigcup_{n\in\omega} s_{2n} \in \mathrm{split}(T)$, then the condition $(S_s, T_t) \leq (S,T)$ and $\mathrm{dom}(\mathrm{stem}(S_s)) = \mathrm{dom}(\mathrm{stem}(S_s))$.

It is missing to prove that we can find oscillation points densely, let $(S,T) \in \mathbb{Q}$ with $\sigma_0 = \mathrm{stem}(S)$ and $\tau_0 = \mathrm{stem}(T)$ and $\delta = \mathrm{dom}(\sigma_0) = \mathrm{dom}(\tau_0)$. Choose $i_0, j_0 < \kappa$ such that $\sigma_0^\frown i_0 \in \mathrm{succ}_S(\sigma_0)$ and i_0 is the minimum with this property, also $\tau_0^\frown j_0 \in \mathrm{succ}_T(\tau_0)$ and $j_0 > \sup(\{i_0\} \cup \sigma_0(l) : l < \delta)$. Inductively, suppose that we have already constructed σ_n, τ_n in $\mathrm{split}(S), \mathrm{split}(T)$ respectively and i_n and j_n. Take then σ_{n+1} and τ_{n+1} to be splitting nodes extending $\sigma_n^\frown i_n$ and $\tau^\frown j_n$ respectively and such that $\sigma_{n+1} \in \mathrm{split}_{\alpha_{n+1}}(S)$ and $\tau_{n+1} \in \mathrm{split}_{\beta_{n+1}}(S)$.

Finally, take i_{n+1} and j_{n+1} such that $i_{n+1} \in \mathrm{succ}_S(\sigma_{n+1})$ and i_{n+1} is the minimum with this property, also $\tau_{n+1}{}^\frown j_{n+1} \in \mathrm{succ}_T(\tau_{n+1})$ and $j_{n+1} > \sup(\{i_{n+1}\} \cup \sigma_{n+1}(l) : l \in \mathrm{dom}(\sigma_{n+1}) \setminus \mathrm{dom}(\sigma_n))$. It is clear from the construction that at the end $\sigma = \bigcup_{n\in\omega} \sigma_n$ and $\tau = \bigcup_{n\in\omega} \tau_n$ are splitting nodes in S and T respectively which belong to the same splitting level, namely to the $\delta^* = \sup_{n\in\omega} \alpha_n = \sup_{n\in\omega} \beta_n$ and we have that for all $i \in \mathrm{dom}(\sigma) \setminus \mathrm{dom}(\tau)$, $\sigma(\gamma) < \tau(\gamma)$, from this it is easy to construct an oscillation point by choosing $i, j < \kappa$ such that $\sigma^\frown i \in \mathrm{succ}_S(\sigma)$, $\tau^\frown j \in \mathrm{succ}_T(\tau)$ and $j < i$. Thus we can refine our condition to have as stem the next splitting node in S and T above $\sigma^\frown i$ and $\tau^\frown j$.

□

Now, let $\{\gamma_\alpha : \alpha < \kappa\}$ be an enumeration of C, define $c : \kappa \to 2$ by:

$$c(\alpha) = \begin{cases} 0 & \text{if } m_0(\gamma_\alpha) \leq m_1(\gamma_\alpha) \\ 1 & \text{if } m_0(\gamma_\alpha) > m_1(\gamma_\alpha) \end{cases}$$

We argue that c is κ-Cohen generic. Let $(S, T) \in \mathbb{Q}$ and $D \subseteq \mathbb{C}_\kappa$ dense, we shall find $(S', T') \leq (S, T)$ and $w \in D$ such that $(S', T') \Vdash w \subseteq \dot{c}$. Let $\sigma = \text{stem}(S)$ and $\tau = \text{stem}(T)$, define the oscillation points between these two partial functions (remember that we can assume that they have the same domain). It is possible then to define a partial function $\rho \in 2^{<\kappa}$ whose domain correspond to the set of limit oscillation points of σ and τ, take then $u \in D$ such that $w \supseteq \rho$ and notice that if we look at the size λ of the set $\text{dom}(w) \setminus \text{dom}(\rho)$; following the same procedure of the lemma above we can thin out our trees S and T to S' and T' in such a way that we have λ-many oscillation points between $\sigma' = \text{stem}(S')$ and $\tau' = \text{stem}(T')$ and such that $(S', T') \Vdash w \subseteq \dot{c}$. □

It is well-known that the full product of countably many Cohen reals collapses the continuum to ω and thus, by [VW98] the same is true for the full product of countably many Miller reals. For strongly inaccessible κ, the situation is different, the κ-support product of κ-Cohen forcing preserves κ^+ [Fri14, Proposition 24] and therefore, under GCH, all cardinals. We shall see below (Proposition 1.102) the same is true for κ-Miller forcing so that we may actually consider the product.

Let $\mathcal{F}$ be a κ-complete normal filter on κ.

Definition 1.100. For a set A of ordinals, $\mathbb{MI}^{\mathcal{F}}_{\kappa,A}$ is the *κ-support product* of $\mathbb{MI}^{\mathcal{F}}_{\kappa}$ with index set A, that is, $\mathbb{MI}^{\mathcal{F}}_{\kappa,A}$ consists of all functions $p : A \to \mathbb{MI}^{\mathcal{F}}_{\kappa}$ such that $\text{supp}(p) = \{\beta \in A : p(\beta) \neq \kappa^{<\kappa}\}$ has size at most κ. $\mathbb{MI}^{\mathcal{F}}_{\kappa,A}$ is ordered coordinatewise: $q \leq p$ if $q(\beta) \leq p(\beta)$ for all $\beta \in A$.

As in the κ-Sacks case, given $\beta \in A$, $\mathbb{MI}^{\mathcal{F}}_{\kappa,A}$ adds a $\mathbb{MI}^{\mathcal{F}}_{\kappa}$-generic function m_β over the ground model. This forcing notion is $<\kappa$-closed and, assuming $2^\kappa = \kappa^+$, has the κ^{++}-cc. For every $F \subseteq A$ of size $< \kappa$ and $\alpha < \kappa$, we define the *fusion ordering* $\leq_{F,\alpha}$ as follows: $q \leq_{F,\alpha} p$ if $q \leq p$ and for every $\beta \in F$, $q(\beta) \leq_\alpha p(\beta)$.

Definition 1.101 (Generalized Miller fusion)**.** $(p_\alpha, F_\alpha : \alpha < \kappa)$ is a *generalized fusion sequence* if $p_\alpha \in \mathbb{MI}^{\mathcal{F}}_{\kappa,A}$, $F_\alpha \in [A]^{<\kappa}$, and

1. $p_{\alpha+1} \leq_{F_\alpha,\alpha} p_\alpha$ and $p_\delta = \bigwedge_{\alpha<\delta} p_\alpha$ when δ is a limit ordinal $<\kappa$,
2. $F_\alpha \subseteq F_{\alpha+1}$, $F_\delta = \bigcup_{\alpha<\delta} F_\alpha$ for limit $\delta < \kappa$ and $\bigcup_{\alpha<\kappa} F_\alpha = \bigcup_{\alpha<\kappa} \text{supp}(p_\alpha)$.

By the analogue of the Generalized Fusion Lemma from Kanamori [Kan80] we see that given such a generalized fusion sequence $(p_\alpha, F_\alpha : \alpha < \kappa)$ the *fusion* $p = \bigwedge_{\alpha<\kappa} p_\alpha$ is a condition in $\mathbb{MI}^{\mathcal{F}}_{\kappa,A}$. This allows us to ensure the preservation of κ^+.

Proposition 1.102. $\mathbb{MI}^{\mathcal{F}}_{\kappa,A}$ preserves κ^+.

Proof. We first fix some notation. Let $\alpha < \kappa$, $T \in \mathbb{MI}^{\mathcal{F}}_{\kappa}$ and $u \in \mathrm{split}_\alpha(T)$. Define T^α_u as follows: if u is a final element of $\mathrm{split}_\alpha(T)$, i.e., $\{v \in \mathrm{split}(T) : v \subsetneq u\}$ has order type exactly α, then $T^\alpha_u = T_u$; otherwise $T^\alpha_u = \{t \in T_u : t \subseteq u$ or $t(\ell(u)) \cap \succ_T (u)$ has order type $> \alpha\}$. Next, for $q \in \mathbb{MI}^{\mathcal{F}}_{\kappa,A}$, $F \subseteq A$, and $\bar{u} = (u_\beta : \beta \in F)$, define $q^\alpha_{\bar{u}}$ by letting

$$q^\alpha_{\bar{u}}(\beta) = \begin{cases} (q(\beta))^\alpha_{u_\beta} & \text{if } \beta \in F \\ q(\beta) & \text{otherwise.} \end{cases}$$

Now assume $p \in \mathbb{MI}^{\mathcal{F}}_{\kappa,A}$ and $\dot{f}$ is an $\mathbb{MI}^{\mathcal{F}}_{\kappa,A}$-name for a function from κ to the ordinals. We recursively construct a generalized fusion sequence $(p_\alpha, F_\alpha : \alpha < \kappa)$ and sets B_α of ordinals such that

- $|F_\alpha| \leq |\alpha|$,
- $|B_\alpha| \leq \max(2^{|\alpha|}, \omega)$,
- for all $\zeta < \alpha$ and all sequences $\bar{u} = (u_\beta : \beta \in F_\alpha)$ with all $u_\beta \in \mathrm{split}_\alpha(p_\alpha(\beta))$, if there is $q \leq_{F_\alpha,0} (p_{\alpha+1})^\alpha_{\bar{u}}$ forcing a value to $\dot{f}(\zeta)$, then for some $\xi \in B_\alpha$, $(p_{\alpha+1})^\alpha_{\bar{u}} \Vdash \dot{f}(\zeta) = \xi$.

For the basic step, let $p_0 = p$ and $B_0 = \emptyset$.

For the successor step, suppose that p_α and F_α have already been constructed. Let $\bar{U} = \{\bar{u} = (u_\beta : \beta \in F_\alpha) : u_\beta \in \mathrm{split}_\alpha(p_\alpha(\beta))$ for all $\beta \in F_\alpha\}$. Since $|\mathrm{split}_\alpha(p_\alpha(\beta))| \leq |(\alpha+1)^{\alpha+1}|$ for all β and $|F_\alpha| \leq |\alpha|$, we see that $|\bar{U}| = 2^{|\alpha|}$ for infinite α. Let $((\bar{u}_\gamma, \zeta_\gamma) : \gamma < \lambda_\alpha)$ enumerate all pairs $(\bar{u}, \zeta) \in \bar{U} \times \alpha$ where $\lambda_\alpha = |\bar{U} \times \alpha| \leq \max(2^{|\alpha|}, \omega)$. Recursively construct a decreasing chain $(q^\gamma_\alpha : \gamma < \lambda_\alpha)$ of conditions and ordinals $(\xi^\gamma_\alpha : \gamma < \lambda_\alpha)$ such that

- $q^0_\alpha = p_\alpha$, $q^\delta_\alpha \leq_{F_\alpha,\alpha} q^\gamma_\alpha$ for all $\gamma \leq \delta$,
- if there is $q \leq_{F_\alpha,0} (q^\gamma_\alpha)^\alpha_{\bar{u}_\gamma}$ forcing a value to $\dot{f}(\zeta_\gamma)$, then $(q^{\gamma+1}_\alpha)^\alpha_{\bar{u}_\gamma} \Vdash \dot{f}(\zeta_\gamma) = \xi^\gamma_\alpha$,
- $q^\delta_\alpha = \bigwedge_{\gamma<\delta} q^\gamma_\alpha$ for limit ordinals δ.

Since the basic step and the limit step are straightforward, it suffices to do the successor step of this recursion. Assume q^γ_α has been produced. If no q as in the second clause exists, let $\xi^\gamma_\alpha = 0$ and $q^{\gamma+1}_\alpha = q^\gamma_\alpha$. If such a q exists, say $q \Vdash \dot{f}(\zeta_\gamma) = \xi$, then let $\xi^\gamma_\alpha = \xi$ and

$$q^{\gamma+1}_\alpha(\beta) = \begin{cases} q(\beta) \cup \bigcup\{(q^\gamma_\alpha(\beta))_u : u \in \mathrm{split}_\alpha(p_\alpha(\beta)) \setminus \{(u_\gamma)_\beta\}\} & \text{if } \beta \in F_\alpha \\ q(\beta) & \text{otherwise.} \end{cases} \quad 0$$

Clearly $q(\beta) = (q^{\gamma+1}_\alpha(\beta))^\alpha_{(u_\gamma)_\beta}$ for $\beta \in F_\alpha$ and $q^{\gamma+1}_\alpha \leq_{F_\alpha,\alpha} q^\gamma_\alpha$. This completes the recursive construction. Let $p_{\alpha+1} = \bigwedge_{\gamma<\lambda_\alpha} q^\gamma_\alpha$ and $B_\alpha = \{\xi^\gamma_\alpha : \gamma < \lambda_\alpha\}$. Clearly, $p_{\alpha+1} \leq_{F_\alpha,\alpha} p_\alpha$ and B_α has size at most λ_α. Finally define $F_{\alpha+1}$ by adding a single element to F_α and by guaranteeing via a book-keeping argument that the union of the F_α will agree with the union of the $\mathrm{supp}(p_\alpha)$.

For the limit step, suppose we already have constructed $(p_\alpha, F_\alpha : \alpha < \delta)$ where δ is a limit ordinal. We let $p_\delta = \bigwedge_{\alpha<\delta} p_\alpha$ and $F_\delta = \bigcup_{\alpha<\delta} F_\alpha$.

Take the fusion $q = \bigwedge_{\alpha<\kappa} p_\alpha$ of the sequence $(p_\alpha, F_\alpha : \alpha < \kappa)$. Then $q \leq p_\alpha$ for all α. We claim that q forces $\operatorname{ran}(\dot{f}) \subseteq B$ where $B = \bigcup_{\alpha<\kappa} B_\alpha$.

To see this, let $r \leq q$ and assume $r \Vdash \dot{f}(\zeta) = \eta$ for some $\zeta < \kappa$ and some η. Choose $\alpha > \zeta$ such that $u_\beta := \operatorname{stem}(r(\beta)) \in \operatorname{split}_\alpha(q(\beta)) = \operatorname{split}_\alpha(p_\alpha(\beta))$ for all $\beta \in F_\alpha$. (This is clearly possible because the sequence $(F_\alpha : \alpha < \kappa)$ is continuous.) Let $\bar{u} = (u_\beta : \beta \in F_\alpha)$. By construction we must have $(p_{\alpha+1})^\alpha_{\bar{u}} \Vdash \dot{f}(\zeta) = \xi$ for some $\xi \in B_\alpha$. Since $r \leq_{F_\alpha,0} q_{\bar{u}} \leq_{F_\alpha,0} (p_{\alpha+1})_{\bar{u}}$, r and $(p_{\alpha+1})^\alpha_{\bar{u}}$ are compatible and therefore $\eta = \xi \in B_\alpha \subseteq B$, as required. $\square$

If we consider the product when $A = \lambda$ we obtain a model in which the cardinal invariants assume the same values as in the κ-Cohen extension (see Proposition 1.50).

Proposition 1.103. Assume $2^\kappa = \kappa^+$ and let $\lambda > \kappa^+$ be a cardinal with $\lambda^\kappa = \lambda$. Then, in the $\mathbb{MI}^{\mathcal{F}}_{\kappa,\lambda}$-generic extension, $\operatorname{non}\mathcal{M}(\kappa) = \kappa^+$ and $\operatorname{cov}\mathcal{M}(\kappa) = 2^\kappa = \lambda$ holds.

Proof. The equality $2^\kappa = \lambda$ follows from $\lambda^\kappa = \lambda$. Using Proposition 1.98 we know that the product $\mathbb{MI}^{\mathcal{F}}_{\kappa,\lambda}$ adds λ-many κ-Cohen functions.

For $\gamma < \delta < \lambda$, let $\dot{c}_{\gamma,\delta}$ be the κ-Cohen function constructed from the κ-Miller functions $\dot{m}_\gamma$ and $\dot{m}_\delta$. Let $\dot{f}$ be a $\mathbb{MI}^{\mathcal{F}}_{\kappa,\lambda}$-name for a function $\dot{f} : 2^{<\kappa} \to 2^{<\kappa}$ with $\sigma \subseteq \dot{f}(\sigma)$, for all $\sigma \in 2^{<\kappa}$. Assume that a condition p forces that $\dot{f}$ is already added by the sub-forcing $\mathbb{MI}^{\mathcal{F}}_{\kappa,B}$ for some $B \subseteq \lambda$. Also assume $\gamma, \delta \notin B$. Then, as in the proof of Proposition 1.50, we can show that $p \Vdash \dot{c}_{\gamma,\delta} \notin A_{\dot{f}}$ where $A_{\dot{f}}$ is defined as in Subsection 4.1.

Let $\mu < \lambda$ and let $(\dot{f}_\beta : \beta < \mu)$ be $\mathbb{MI}^{\mathcal{F}}_{\kappa,\lambda}$-names for functions from $2^{<\kappa}$ to $2^{<\kappa}$ with $\sigma \subseteq \dot{f}_\beta(\sigma)$ for all $\sigma \in 2^{<\kappa}$ and $\beta < \mu$. By the κ^{++}-cc we can find sets $B_\beta \subseteq \lambda$, for all $\beta < \mu$, of size at most κ^+ such that $\dot{f}_\beta$ is added by the sub-forcing $\mathbb{MI}^{\mathcal{F}}_{\kappa,B_\beta}$. Hence, if $(\gamma, \delta) \notin \bigcup_{\beta<\mu} B_\beta$, we have $\Vdash \dot{c}_{\gamma,\delta} \notin \bigcup_{\beta<\mu} A_{\dot{f}_\beta}$, and $\operatorname{cov}\mathcal{M}(\kappa) \geq \lambda$ follows.

For $\operatorname{non}\mathcal{M}(\kappa) \leq \kappa^+$ it is enough to see that the set of the first κ^+ many κ-Cohen functions $\{\dot{c}_{\gamma,\delta} : \gamma < \delta < \kappa^+\}$ is non-meager in the generic extension. Fix $\dot{f}_\beta$, $\beta < \kappa$, as before. Also let $p \in \mathbb{MI}^{\mathcal{F}}_{\kappa,\lambda}$ be arbitrary. By the proof of the previous proposition, we can find $q \leq p$ and $B \subseteq \lambda$ of size $\leq \kappa$ such that q forces all $\dot{f}_\beta$ are already added by the sub-forcing $\mathbb{MI}^{\mathcal{F}}_{\kappa,B}$. Thus q forces that $\dot{c}_{\gamma,\delta}$ is not contained in the union of the $A_{\dot{f}_\beta}$, as required. $\square$

1.4. Open questions

The study of the generalized Baire spaces is relatively new and right now there are a bunch of open questions that involve topics like *Generalized descriptive set theory* and also *generalized cardinal invariants*. We refer the reader to the article *Questions on generalised Baire spaces* [Kho+16] to find some of the current directions of research.

Although we managed to obtain a good approximation of the diagram and to prove some consistency results there are a lot of open questions we still have not solved and now we address:

1. Assume κ is strongly inaccessible (or even supercompact). Is $\mathfrak{b}(\kappa) < \operatorname{non}\mathcal{M}(\kappa)$ consistent?
2. (Matet, Shelah [MS12, Section 4], see also [Kho+16, Question 3.8, part 1]) Is it consistent that κ is a successor cardinal and $\operatorname{cv}(\kappa) < \mathfrak{d}(\kappa)$?
3. (see also [MS12, Section 4]) Assume $\kappa = 2^{<\kappa}$ is a successor cardinal. Is $\mathfrak{b}(\kappa) = \operatorname{non}\mathcal{M}(\kappa)$ and $\mathfrak{d}(\kappa) = \operatorname{cov}\mathcal{M}(\kappa)$?
4. Assume κ is strongly inaccessible. Are $\mathfrak{b}_{\mathrm{id}}(\in^*_p)(\kappa) < \operatorname{add}\mathcal{M}(\kappa)$ and $\operatorname{cof}\mathcal{M}(\kappa) < \mathfrak{d}_{\mathrm{id}}(\in^*_p)(\kappa)$ consistent?
5. Is $\mathfrak{b}_{\mathrm{id}}(\in^*)(\kappa) < \mathfrak{b}_h(\in^*)(\kappa)$ consistent where h is the power set function?
6. Is it consistent that three cardinals of the form $\mathfrak{d}_h(\in^*)(\kappa)$ for different $h \in \kappa^\kappa$ simultaneously assume distinct values?
7. Is it consistent that for some function g strictly dominating the power set function h, $\mathfrak{d}_g(\in^*)(\kappa) < \mathfrak{d}_h(\in^*)(\kappa)$ is consistent?
8. For $h \in \kappa^\kappa$ with $\sup_{\alpha<\kappa} h(\alpha) = \kappa$. Is the h-generalized Laver property preserved under κ-support iterations?
9. More specifically. Let h be the power set function. Assume κ is an indestructible supercompact cardinal. Does the κ-support iteration of forcings of type $\mathbb{MI}^\kappa_\mathcal{U}$ have the generalized h-Laver property?
10. Is it consistent for strongly inaccessible (or even supercompact) κ that $\operatorname{add}\mathcal{M}(\kappa) < \mathfrak{b}_\kappa$? That $\mathfrak{d}(\kappa) < \operatorname{cof}\mathcal{M}(\kappa)$?

Chapter 2

The generalized ultrafilter number

The results of this chapter are joint work with Andrew-Brooke Taylor, Vera Fischer and Sy-David Friedman and can be also founded in [Fis+17]. In [DS03] Džamonja and Shelah construct a model with a universal graph at the successor of a strong limit singular cardinal of countable cofinality. Afterwards Garti and Shelah pointed out [Claim 2.3 [GS12]] that a variant of such model model witnesses the consistency of the inequality $\mathfrak{u}(\kappa) = \kappa^+ < 2^\kappa$. See also [Bro].

This chapter presents a modification of the forcing construction used by Džamonja and Shelah, which gives us a cardinal preserving generic extension in which $\mathfrak{u}(\kappa) = \kappa^*$ when κ is a supercompact cardinal and $\kappa^* > \kappa$ is a regular cardinal while 2^κ is $> \kappa^*$. The idea of this construction originates in the class notes from Friedman [Fri14] about cardinal invariants for uncountable cardinals which provide a result that states that if after our iteration κ is still supercompact and we take a normal measure $\mathcal{U}$ on κ in the final extension, then there is a set of ordinals of order type κ^* such that the restrictions of $\mathcal{U}$ to the corresponding intermediate extensions coincide with ultrafilters which have been chosen generically (see Lemma 2.17). Furthermore, our construction allows us to decide the values of many of the higher analogs of the known classical cardinal characteristics of the continuum, including some of the ones defined in 1; by interleaving arbitrary κ-directed closed posets cofinally in the iteration.

Definition 2.1. Let κ be a regular cardinal $\geq \omega$. A *filter* (on κ) is a family $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ that consist of unbounded subsets of κ (we are interested just in non-principal filters) and in addition satisfies $\emptyset \notin \mathcal{F}$, $X \in \mathcal{F}$ and it is closed under finite intersections and supersets. An *ultrafilter* is a maximal filter with respect to the inclusion $\subseteq$ relation. Finally, a *base* $\mathcal{B}$ for $\mathcal{F}$ is a subfamily of $\mathcal{F}$ such that, for all $X \in \mathcal{F}$ there exists $Y \in \mathcal{B}$ with $Y \subseteq^* X$, i.e. $|Y \setminus X| < \kappa$.

Definition 2.2 (The ultrafilter number)**.**

$\mathfrak{u}(\kappa) = \min\{|\mathcal{B}|\colon \mathcal{B}$ is an base for a uniform ultrafilter on $\kappa\}$.[1]

Here *uniform* means that all the sets in the ultrafilter have size κ.

[1]Of course, this definition works for every cardinal κ.

2.1. A word on the countable case

The ultrafilter number on the countable case has been widely studied. Some results involving this cardinal can be founded for example in [Bla10]. The purpose of this section is to give the necessary motivation for the result in Theorem 2.43, we provide a review of the analogous results in the countable case, offering, therefore, the possibility to compare both constructions and motivation for the definitions presented in the upcoming sections of this chapter.

The following proposition establishes some simple cardinal bounds for $\mathfrak{u}$:

Proposition 2.3. It is ZFC provable that (see Blass [Bla10]):

- $\aleph_1 \leq \mathfrak{u} \leq \mathfrak{c}$.
- One of its lower bounds is the cardinal $\mathfrak{r}$, and as consequence $\mathfrak{b}$, $\mathfrak{e}$, $\mathfrak{h}$, $\mathfrak{t}$ and $\mathfrak{p}$.

Proof. We prove $\mathfrak{r} \leq \mathfrak{u}$ from which $\aleph_1 \leq \mathfrak{u}$ follows (for $\mathfrak{r} \geq \aleph_1$ see [Bla10]). Given an ultrafilter $\mathcal{U}$ with a base $\mathcal{B}$ we notice that this is also an unsplittable family of subsets of ω. Suppose towards a contradiction that there is a set $X \subseteq \omega$ that splits every element of $\mathcal{B}$, since $\mathcal{U}$ is ultra we should have that either X or $\omega \setminus X \in \mathcal{U}$. In the first case we can also find $F \in \mathcal{B}$ such that $F \subseteq X$, but then clearly X does not split F. The other case is analogous. $\square$

Figure 3.1 shows the neighbors of the ultrafilter number for $\kappa = \omega$, as well as the provable ZFC relations between them (see [Bla10] for more details):

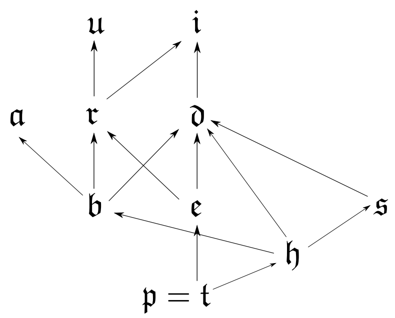

Figure 2.1.: $\mathfrak{u}$ and its neighbors

As usual, once one has all the obvious ZFC inequalities, consistency related questions arise. For instance, the most simple one is whether it is possible to separate the continuum from the ultrafilter number $\mathfrak{u}$. The answer to this question is positive and its answer is due to Kunen.

Theorem 2.4 (Kunen, Lemma V.4.27 in [Kun80])**.** *It is consistent that* $\mathfrak{u} = \aleph_1$ *and* $\mathfrak{c} = \kappa$ *for* $\kappa > \aleph_1$.

Definition 2.5 (Mathias-Příkrý Forcing)**.** Let $\mathcal{U}$ be an ultrafilter on ω. Mathias-Příkrý forcing $\mathbb{M}_{\mathcal{U}}$ has, as its set of conditions $\{(s, A) : s \in [\omega]^{<\omega}$ and $A \in \mathcal{U}\}$, ordered by: $(t, B) \leq (s, A)$ if and only if $t \supseteq s, B \subseteq A$ and $t \setminus s \subseteq A$.

Observation 2.6. The generic real added by Mathias-Příkrý forcing $\mathbb{M}_{\mathcal{U}}$ over a model V given by $m_{\mathcal{U}} = \bigcup\{s : \exists A \in \mathcal{U}(s, A) \in G\}$, where G is $\mathbb{M}_{\mathcal{U}}$-generic has the following property: for all $X \in \mathcal{U} \cap V$, $m_{\mathcal{U}} \subseteq^* X$ (we say that $m_{\mathcal{U}}$ is a *pseudointersection* of $\mathcal{U}$). Just notice that, given $X \in \mathcal{U} \cap V$ the set $\mathcal{D}_X = \{(s, A) \in \mathbb{M}_{\mathcal{U}} : X \subseteq A\}$ is dense.

Proof of Theorem 2.4. Start with a ground model V in which $\mathfrak{c} = \kappa$ (to achieve this add, to a model of GCH κ-many Cohen reals), the final model is obtained from a finite support iteration $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \aleph_1, \beta < \aleph_1)$ over V of Mathias forcing relative to some ultrafilters that are constructed along the iteration.

Recall that, if at step $\alpha < \omega_1$ Mathias forcing respect to an ultrafilter $\mathcal{U}_\alpha$ in $V^{\mathbb{P}_\alpha}$ is used (let $\dot{\mathcal{U}}_\alpha$ be a $\mathbb{P}_\alpha$-name for it), we add generically a subset of ω, $\dot{m}_\alpha$ that is a pseudointersection of the ultrafilter $\dot{\mathcal{U}}_\alpha$, i.e. in $V^{\mathbb{P}}$ we have that for all $F \in \mathcal{U}_\alpha$, $\dot{m}_\alpha \subseteq^* F$.

Thus, define $\mathbb{P}_0 = \mathbb{1}$ and $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \mathbb{M}(\dot{\mathcal{U}}_\alpha)$ where $\dot{\mathcal{U}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a non principal ultrafilter that satisfies both $\dot{m}_\alpha \in \dot{\mathcal{U}}_{\alpha+1}$ and $\forall \beta < \alpha, \Vdash_\alpha \dot{\mathcal{U}}_\beta \subseteq \dot{\mathcal{U}}_\alpha$. The sequence of desired ultrafilter should be chosen as follows: start in the ground model V with any ultrafilter $\mathcal{U}_0 \in V$; for the successor step note that in $V^{\mathbb{P}_\alpha}$ the set $\mathcal{U}_\alpha \cup \{\dot{m}_\alpha\}$ generates a filter, which using Zorn's lemma (in $V^{\mathbb{P}_\alpha}$) can be extended to an ultrafilter $\mathcal{U}_{\alpha+1}$. Finally in the limit steps $\alpha < \omega_1$, extend the filter generated by $\bigcup_{\beta<\alpha} \dot{\mathcal{U}}_\beta$ to an ultrafilter $\mathcal{U}_\alpha$.

At the end the inequality $\mathfrak{u} \leq \aleph_1$ is witnessed by the ultrafilter $\mathcal{U} = \bigcup_{\alpha<\omega_1} \dot{\mathcal{U}}_\alpha$, which is generated by the sets $\{\dot{m}_\alpha : \alpha < \aleph_1\}$. To finish note that the chain condition guarantees that $\mathfrak{c} = \kappa$ still holds in $V^{\mathbb{P}}$. $\square$

More sophisticated results involving the ultrafilter number on ω can be found in the literature; for example Shelah developed a method to construct a maximal independent family that can be preserved upon some forcing notion increasing $\mathfrak{u}$, and so he obtained a generic extension in which $\mathfrak{i} < \mathfrak{u}$ (see [She92]). Our motivation, however, goes in the direction of generalizing Theorem 2.4 for uncountable cardinals.

2.2. The model for the uncountable case

In this section, we present first a model where the inequality $\kappa^+ < \mathfrak{u}(\kappa) < 2^\kappa$ holds, and then we will notice that the construction can be enhanced such that we can also decide other cardinal characteristics.

The first attempt to obtain such model is to generalize the construction in Theorem 2.4. However, this does not work: the first obstruction when generalizing the proof above for the countable case (i.e. performing a $<\kappa$-support iteration of a κ-ultrafilter version of Mathias-Příkrý forcing) is the following: Note that, if $(\mathcal{U}_n : n \in \omega)$ is an increasing sequence of κ-complete ultrafilters, the union $\bigcup_{n\in\omega} \mathcal{U}_n$ may not even be a κ-complete filter. Hence, if we would like to use a suitable generalization of Mathias-Příkrý forcing and iterate it, our construction will have serious problems in steps of cofinality less than κ.

Why κ-complete ultrafilters?: In the countable case, Mathias-Příkrý forcing is ccc (moreover σ-centered) and this property guarantees that we preserve cardinals $\geq \aleph_1$. Now, in our case we also want to preserve cardinals but now it is a bit harder. Namely, we would like that our forcing will have some κ^+-chain condition, so we preserve cardinals $\geq \kappa^+$, but in addition we should also preserve cardinals $\leq \kappa$. In order to do this we can use a κ-closed forcing notion. This motivates the following definition:

Definition 2.7 (Generalized Mathias forcing)**.** Let κ be a measurable cardinal and $\mathcal{F}$ be a κ-complete filter on κ. The Generalized Mathias Forcing $\mathbb{M}^\kappa_{\mathcal{F}}$ has, as its set of conditions, $\{(s, A) : s \in [\kappa]^{<\kappa}$ and $A \in \mathcal{F}\}$, and the ordering given by $(t, B) \leq (s, A)$ if and only if $t \supseteq s, B \subseteq A$ and $t \setminus s \subseteq A$. We denote by $\mathbb{1}_{\mathcal{F}}$ the maximum element of $\mathbb{M}^\kappa_{\mathcal{F}}$, that is $\mathbb{1}_{\mathcal{F}} = (\emptyset, \kappa)$.

As in the countable case, if $\mathcal{F}$ is a κ-complete filter $\mathbb{M}^\kappa_{\mathcal{F}}$ adds generically an unbounded set $m^\kappa_{\mathcal{F}} \subseteq \kappa$ that has also the property $m^\kappa_{\mathcal{F}} \subseteq^* F$ for all $F \in \mathcal{F}$, also this forcing is κ-centered, so κ^+-cc and κ-closed as desired.

Now, say we want to iterate this forcing with $<\kappa$-support and length some ordinal $\lambda > \kappa^+$. The first concern is that we want to have κ-complete ultrafilters to iterate with (which is not at all trivial in this context), so of course we are already at the level of measurability.

It is also well-known that if we have a large cardinal property, this property can be destroyed after forcing. That is the reason why we will start with κ to be a supercompact cardinal and use the famous Laver preparation, to preserve this property after forcing.

2.2.1. Laver preparation

Recall that a cardinal κ is *supercompact* if, for all $\lambda > \kappa$ there are elementary embeddings $j : V \to M$ such that $\text{crit}(j) = \kappa$, the model M is closed under λ-sequences (write $M^\lambda \subseteq M$) and $j(\kappa) > \lambda$. If κ is supercompact, Laver preparations gives us a tool, to preserve the supercompactness of κ after forcing with $< \kappa$-directed closed forcings. The construction is due to Laver and we give here all the details for self-containing purposes.

Theorem 2.8 (Laver [Lav78])**.** *If κ is supercompact, then there is a κ^+-cc forcing notion $\mathbb{Q}$ of size κ, such that in $V^{\mathbb{Q}}$, κ is supercompact and remains supercompact upon forcing with any κ-directed closed partial ordering.*

The proof of the theorem needs the following lemma, which assures the existence of the well-known Laver diamonds.

Lemma 2.9. Let κ be supercompact. Then there is a function $h : \kappa \to V_\kappa$ such that for every x and every $\lambda \geq \kappa$ such that $x \in H_{\lambda^+}$, there is a $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$, $M^\lambda \subseteq M$ and $j(f)(\kappa) = x$. h is called a *Laver diamond* for κ.

Proof. Suppose towards a contradiction that for each function $h : \kappa \to V_\kappa$ there exist $\lambda_h \geq \kappa$ minimal such that some $x \in H_{\lambda_h^+}$ the pair (λ_h, x) witnesses that h is not a Laver diamond for κ. Let ν be greater than all the λ_f and use the supercompactness from κ to find an elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \nu$ and M is closed under sequences of length ν.

Now, we define a function $h : \kappa \to V_\kappa$ as follows: Suppose that we have inductively constructed $h \restriction \alpha$ and consider the two following cases:

- If $h \restriction \alpha = h_\alpha$ is not a Laver diamond for α where (λ_{h_α}, y) witness this property with λ_{h_α} minimal respect to this property, define $h(\alpha)$ to be y.
- Otherwise, $h(\alpha) = 0$.

Let also X to be the set of $\alpha < \kappa$ such that $h(\alpha) \neq 0$, by elementarity and the fact that h is not a Laver diamond for κ, we have that $\kappa \in j(X)$. Here $j(X)$ is the set of ordinals $\alpha < j(\kappa)$ such that $j(h)(\alpha) \neq 0$.

Put $j(h)(\kappa) = x$, then $M \models j(h) \restriction \kappa = h$ is not a Laver diamond for κ, so in M (λ_h^M, x) witnesses this property. Note that also $\lambda_h^M = \lambda_f$, because $M^\nu \subseteq M$. But then, the same pair actually witnesses that h is not a Laver function for κ in V, this is clearly a contradiction because by our assumptions $j(\kappa) > \lambda_h$ and $j(h)(\kappa) = x$. $\square$

Proof of Theorem 2.8. Let $h : \kappa \to V_\kappa$ be a Laver diamond for κ. The forcing notion $\mathbb{Q}$ will be obtained as a reverse Easton iteration of length κ. We define inductively the posets $\mathbb{Q}_\alpha$ and simultaneously ordinals λ_α for all $\alpha < \kappa$ as follows:

- At limit stages $\alpha < \kappa$, $\mathbb{Q}_\alpha$ has all the sequences p in the inverse limit such that for every regular cardinal $\gamma \leq \alpha$, $|\mathrm{supp}(p) \cap \gamma| < \gamma$. Also $\lambda_\alpha = \sup_{\gamma<\alpha} \lambda_\gamma$.
- If we have already constructed $\mathbb{Q}_\alpha$, $\mathbb{Q}_{\alpha+1} = \mathbb{Q}_\alpha * \dot{\mathbb{P}}_\alpha$ where the order $\dot{\mathbb{P}}_\alpha$ is trivial unless $\lambda_\beta < \alpha$, for all $\beta < \alpha$ and $h(\alpha) = (\dot{P}, \lambda)$ where λ is an ordinal and $\dot{P}$ is a $\mathbb{Q}_\alpha$-name for an α-directed closed forcing. In this case, put $\dot{\mathbb{P}}_\alpha = \dot{P}$ and $\lambda_{\alpha+1} = \lambda$.

Let G be a $\mathbb{Q}$-generic filter, it is enough now to prove that if $\mathbb{P} \in V[G]$ is such that $\Vdash_{\mathbb{Q}}$ $\dot{\mathbb{P}}$ is κ-directed closed forcing, then κ is still supercompact after forcing with it. Let λ be a sufficiently large cardinal such that $\dot{\mathbb{P}} \in H(\lambda^+)$, then using that h is a Laver diamond for κ, we can find $j : V \to M$ elementary embedding with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, $M^\lambda \subseteq M$ and $j(h)(\kappa) = \dot{\mathbb{P}}$.

In M, $j(\mathbb{Q}_\alpha : \alpha \leq \kappa)$ is an iteration of length $j(\kappa)$ which, by elementarity and the fact that $\mathrm{crit}(j) = \kappa$ has to have as initial segment the iteration $\mathbb{Q}$. Write then $j(\mathbb{Q}_\alpha : \alpha \leq \kappa) = (\mathbb{Q}_\alpha : \alpha \leq j(\kappa))$ (we refer to this iteration as $\mathbb{Q}_{j(\kappa)}$). Moreover, note that by definition of the forcing $\mathbb{Q}$ and elementarity $\mathbb{Q}_{\kappa+1} = \mathbb{Q} * \dot{\mathbb{P}}$ and for $\kappa < \delta < \lambda$, $\mathbb{Q}_{\delta+1} = \mathbb{Q}_\delta * \mathbb{1}$.

The tail of the iteration $(\mathbb{Q}_\alpha : \lambda \leq \alpha \leq j(\kappa))$ is a $\geq \lambda$-closed forcing notion in M (and hence in V because $M^\lambda \subseteq M$). Also $j(\dot{\mathbb{P}})$ it is forced to be a λ-directed closed forcing notion, thus $\mathbb{Q}_{j(\kappa)} * j(\dot{\mathbb{P}}) = \mathbb{Q} * \dot{\mathbb{P}} * \dot{\mathbb{R}}$ where $\mathbb{R}$ is a λ-directed closed forcing living in the extension $V^{\mathbb{Q}*\dot{\mathbb{P}}}$.

Let H be $\dot{\mathbb{P}}^G$-generic over $V[G]$, it to prove that κ remains supercompact it suffices to extend the elementary embedding $j : V \to M$ to an embedding j^* with from $V[G * H]$ to some generic extension of M, $M[H']$. We describe then, how to construct the $j(\mathbb{Q} * \dot{\mathbb{P}})$-generic H' over M.

Using the argument above we notice that $H' \restriction \kappa = G$ and that $H'(\kappa) = H$ because $H'(\kappa)$ must be $j(h)(\kappa)^G = \dot{\mathbb{P}}$-generic. $H' \restriction (\kappa, j(\kappa))$ will be constructed using that the forcing in this interval has only $|[\lambda]^{<\kappa}{}^V_\kappa| = \lambda^+$-many antichains, is λ^+-directed closed and the model M is closed under λ-sequences.

Finally, we define how to construct the generic H' at stage $j(\kappa)$ making sure that it contains $j'[H]$ where j' is the partial lifting $j' : V[G] \to M[H \restriction j(\kappa)]$. Here we use that the forcing $\dot{\mathbb{P}}$ is κ-directed closed which implies that in

$$M \models \text{“the forcing notion } j(\dot{\mathbb{P}}) \text{ is } j(\kappa)\text{-directed closed at stage } j(\kappa)\text{”}.$$

We also have $j'[H]$ is an element of $M[H \restriction j(\kappa)]$ (which holds by supercompactness) and this is clearly a directed set (it is a *filter*). Hence, we can find a master condition $q \leq j'[H]$. So choosing $H(j(\kappa))$ to contain q, we have extended the embedding completely and so, we have that κ is still supercompact in $V[G * H]$. $\square$

2.2.2. The model

In this section we describe the construction of the model we are aiming for. Assume that we have already used the Laver preparation forcing S_κ over our ground model V. In the extension V^{S_κ} we define the following forcing notion:

Definition 2.10 (The forcing)**.** Let Γ be such that $\Gamma^\kappa = \Gamma$. We will define an iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \Gamma^+, \beta < \Gamma^+ \rangle$ of length Γ^+ recursively as follows:

- *If α is an even ordinal* (abbreviated $\alpha \in$ EVEN), let NUF denote the set of normal ultrafilters on κ in $V^{\mathbb{P}_\alpha}$. Then let $\mathbb{Q}_\alpha$ be the poset with underlying set of conditions $\{\mathbb{1}_{\mathbb{Q}_\alpha}\} \cup \{\{\mathcal{U}\} \times \mathbb{M}^\kappa_{\mathcal{U}} : \mathcal{U} \in \text{NUF}\}$ and extension relation stating that $q \leq p$ if and only if either $p = \mathbb{1}_{\mathbb{Q}_\alpha}$, or there is $\mathcal{U} \in$ NUF such that $p = (\mathcal{U}, p_1)$, $q = (\mathcal{U}, q_1)$ and $q_1 \leq_{\mathbb{M}^\kappa_{\mathcal{U}}} p_1$ (see Figure 2.2.2).
- *If α is an odd ordinal* (abbreviated $\alpha \in$ ODD), let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for a κ-centered, κ-directed closed forcing notion of size at most Γ.

Definition 2.11 (The supports)**.** We define three different kinds of support for conditions $p \in \mathbb{P}_\alpha$ with $\alpha < \Gamma^+$:

- The *Ultrafilter Support* $\mathrm{USup}(p)$ corresponds to the set of ordinals $\beta \in \mathrm{dom}(p) \cap \mathrm{EVEN}$ such that $p \restriction \beta \Vdash_{\mathbb{P}_\beta} p(\beta) \neq \mathbb{1}_{\mathbb{Q}_\beta}$.
- The *Essential Support* $\mathrm{SSup}(p)$ consists of all $\beta \in \mathrm{dom}(p) \cap \mathrm{EVEN}$ such that $\neg(p \restriction \beta \Vdash_{\mathbb{P}_\beta} p(\beta) \in \{\check{\mathbb{1}}_{\mathbb{Q}_\beta}\} \cup \{(\mathcal{U}, \mathbb{1}_\mathcal{U}) : \mathcal{U} \in \mathrm{NUF}\})$ (for the definition of $\mathbb{1}_\mathcal{U}$ see Definition 2.7).
- The *Directed Support* $\mathrm{RSup}(p)$, consists of all $\beta \in \mathrm{dom}(p) \cap \mathrm{ODD}$ such that $\neg(p \restriction \beta \Vdash p(\beta) = \mathbb{1}_{\dot{\mathbb{Q}}_\beta})$.

We require that the conditions in $\mathbb{P}_{\Gamma^+}$ have support bounded below Γ^+ and also that given $p \in \mathbb{P}_{\Gamma^+}$ if $\beta \in \mathrm{USup}(p)$ then for all $\alpha \in \beta \cap \mathrm{EVEN}$, $\alpha \in \mathrm{USup}(p)$. Finally we demand that both $\mathrm{SSup}(p)$ and $\mathrm{RSup}(p)$ have size $< \kappa$ and are contained in $\sup(\mathrm{USup}(p))$, that is $\mathrm{supp}(p)$ (the entire support of p) and $\mathrm{USup}(p)$ have the same supremum.

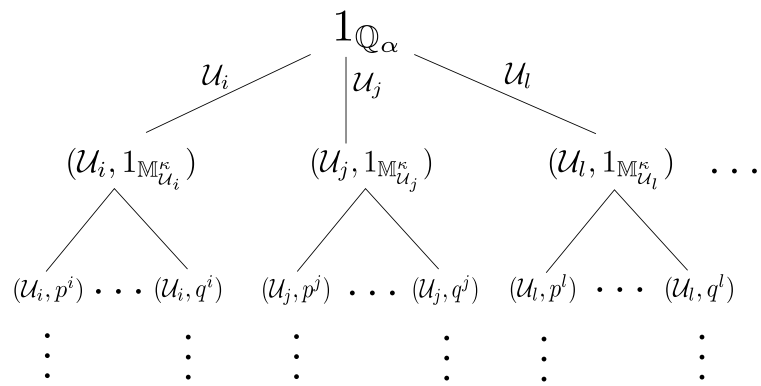

Figure 2.2.: The forcing $\mathbb{Q}_\alpha$ (α even) in the $V^{\mathbb{P}_\alpha}$ extension

Now, we want to ensure that our iteration preserves cardinals. To do so, we prove that the iteration is enough directedly closed and also have a chain condition when we restrict it below some condition. Let $\mathbb{P} := \mathbb{P}_{\Gamma^+}$.

Lemma 2.12. $\mathbb{P}$ is κ-directed closed.

Proof. We know that $\mathbb{M}^\kappa_\mathcal{U}$, as well as all iterands $\mathbb{Q}_\alpha$ for $\alpha \in \mathrm{ODD}$, are κ-directed closed forcings. Take $D = \{p_\alpha : \alpha < \delta < \kappa\}$ a directed set of conditions in $\mathbb{P}$. We want to define a common extension p for all elements in D. First define $\mathrm{dom}(p) = \bigcup_{\alpha<\delta} \mathrm{dom}(p_\alpha)$. For $j \in \mathrm{dom}(p)$ define $p(j)$ by induction on j. We work in $V^{\mathbb{P}_j}$ and assume that $p \restriction j \in \mathbb{P}_j$.

We have the following cases:

- If j is even and $j \notin \bigcup_{\alpha<\delta} \mathrm{SSup}(p_\alpha)$, then using directedness we can find a name $\dot{\mathcal{U}}$ for a normal ultrafilter such that for all $\alpha < \delta$, $p \restriction j \Vdash (\dot{\mathcal{U}}, \check{\mathbb{1}}_\mathcal{U}) \leq p_\alpha(j)$. Define $p(j)$ to be the canonical name for $(\mathcal{U}, \mathbb{1}_\mathcal{U})$.

- If j is even and $j \in \mathrm{SSup}(p_\alpha)$ for some $\alpha < \delta$, then again using directedness it is possible to find a single name $\dot{\mathcal{U}}$ for an ultrafilter such that for $\alpha < \delta$ with $j \in \mathrm{SSup}(p_\alpha)$, $p \restriction j \Vdash p_\alpha(j) \in \{\dot{\mathcal{U}}\} \times \mathbb{M}^{\check{\kappa}}_{\dot{\mathcal{U}}}$, and $\Vdash_{\mathbb{P}_j} \mathbb{M}^{\check{\kappa}}_{\dot{\mathcal{U}}}$ is $\check{\kappa}$-directed closed. We can thus find a condition q such that $p \restriction j \Vdash q \leq p_\alpha(j)$ for all $\alpha < \delta$. Define $p(j) = q$.
- If j is odd, use the fact that in the $\mathbb{P}_j$ extension $\mathbb{Q}_j$ is κ-directed closed on the directed set $X_j = \{p_\alpha(j) : \alpha < \delta < \kappa\}$ to find $p(j)$ a condition stronger than all the ones in X_j.

□

Notation: For any $p \in \mathbb{P}_\beta$, $\beta < \Gamma^+$ let $\mathbb{P}_\beta \downarrow p$ denote the set $\{q \in \mathbb{P}_\beta : q \leq p\}$.

Lemma 2.13. Let $p \in \mathbb{P}_{\Gamma^+}$ and let $i = \sup \mathrm{USup}(p) = \sup \mathrm{Supt}(p)$. Then $\mathbb{P}_i \downarrow (p \restriction i)$ is κ^+-cc and has a dense subset of size at most Γ.

Proof. It is enough to observe that $\mathbb{P}_i \downarrow (p \restriction i)$ is basically a $<\kappa$-support iteration of κ-centered, κ-directed closed forcings of size at most Γ. Then the proof is a straightforward generalization of Lemma V.4.9 – V.4.10 in [Kun80]. □

Lemma 2.14. Let $\{\mathcal{A}_\alpha\}_{\alpha<\Gamma}$ be maximal antichains in $\mathbb{P}$ below $p \in \mathbb{P}$. Let $j^* = \sup \mathrm{Supt}(p)$. Then there is $q \in \mathbb{P}$ such that $q \restriction j^* = p$, $\mathrm{Supt}(q) \setminus \mathrm{Supt}(p) \subseteq \mathrm{USup}(q) \setminus \mathrm{SSup}(q)$ and for all $\alpha < \Gamma$, the set $\mathcal{A}_\alpha \cap (\mathbb{P}_{i^*} \downarrow q)$ is a maximal antichain in $\mathbb{P}_{i^*} \downarrow q$ (and hence in $\mathbb{P} \downarrow q$), where $i^* = \sup \mathrm{Supt}(q)$.

Proof. Let $\bar{\mathbb{P}} := \mathbb{P}_{j^*} \downarrow p$ and let $w \in \bar{\mathbb{P}}$. Then there is a condition r extending both w and an element of $\mathcal{A}_0$ and we can find p_1 such that $p_1 \restriction j^* = p$, $\mathrm{SSup}(p_1) \cup \mathrm{RSup}(p_1) \subseteq j^*$, and $r \in \mathbb{P}_{j_1} \downarrow p_1$, where $j_1 = \sup \mathrm{Supt}(p_1)$. Since $\bar{\mathbb{P}}$ has a dense subset of size at most Γ, in Γ-steps we can find q_0 such that $q_0 \restriction j^* = p$ and every condition in $\bar{\mathbb{P}}$ is compatible with an element of $\mathcal{A}_0 \cap (\mathbb{P}_{j_0^*} \downarrow q_0)$, where $j_0^* = \sup \mathrm{Supt}(q_0)$.

Since we have only Γ many antichains $\{\mathcal{A}_\alpha\}_{\alpha<\Gamma}$, in Γ steps we can obtain the desired condition q. □

Corollary 2.15. If $p \Vdash \dot{X} \subseteq \kappa$ for some $\mathbb{P}$-name $\dot{X}$, then there are $q \leq p$ and $j < \Gamma^+$ such that $\dot{X}$ can be seen as a $\mathbb{P}_j \downarrow q$-name.

Proof. For each $\alpha < \kappa$ fix a maximal antichain $\mathcal{A}_\alpha$ of conditions below p deciding if α belongs to $\dot{X}$. Then, let q be the condition given by Lemma 2.14 and take $j := \sup \mathrm{Supt}(q)$. Then $q \leq p$ and $\dot{X}$ can be seen as a $\mathbb{P}_{\sup \mathrm{Supt}(q)} \downarrow q$-name. □

Corollary 2.16. Let $p \Vdash \dot{f}$ is a $\mathbb{P}$-name for a function from Γ into the ordinals. Then there is a function $g \in V$ and $q \leq p$ such that $q \Vdash \dot{f}(\alpha) \in g(\alpha)$ for $\alpha < \Gamma$ and $|g(\alpha)| \leq \kappa$ for all α. In particular, $\mathbb{P}$ preserves cofinalities and so cardinalities.

Proof. Let $\mathcal{A}_\alpha$ be a maximal antichain of conditions below p deciding a value for $\dot{f}(\alpha)$. Use Lemma 2.14 to find $q \leq p$ such that $\mathcal{A}_\alpha \cap \mathbb{P} \downarrow q$ is a maximal antichain in $\mathbb{P} \downarrow q$ for all $\alpha < \Gamma$. Finally define the function $g \in V$ as follows: $g(\alpha) = \{\beta : \exists r \leq q$ such that $r \Vdash \dot{f}(\alpha) = \beta\}$. □

We now present the key lemmas that will allow us to construct the witness for $\mathfrak{u}(\kappa) = \kappa^*$.

Lemma 2.17 (Main Lemma)**.** Let κ be a supercompact cardinal and κ^* be a cardinal satisfying $\kappa < \kappa^* \leq \Gamma$, κ^* regular. Suppose that $p \in \mathbb{P}$ is such that $p \Vdash \dot{\mathcal{U}}$ is a normal ultrafilter on κ.[2]

Then, for some $\alpha < \Gamma^+$ there is an extension $q \leq p$ such that $q \Vdash \dot{\mathcal{U}}_\alpha = \dot{\mathcal{U}} \cap V[G_\alpha]$. Moreover this can be done for a set of ordinals $S \subseteq \Gamma^+$ of order type κ^* in such a way that $\forall \alpha \in S(\dot{\mathcal{U}} \cap V_\alpha \in V[G_\alpha])$ and $\dot{\mathcal{U}} \cap V[G_{\sup S}] \in V[G_{\sup S}]$. Here $\dot{\mathcal{U}}_\alpha$ is the canonical name for the ultrafilter generically chosen at stage α.

Proof. Let $\alpha_0 = \sup \mathrm{USup}(p)$. Then $\mathbb{P}_{\alpha_0} \downarrow p$ is κ^+-cc and has a dense subset of size at most Γ. Thus there are just Γ-many $\mathbb{P}_{\alpha_0} \downarrow p$-names for subsets of κ. Let $\bar{X} = (\dot{X}_i : i < \Gamma)$ be an enumeration of them.

We view each condition in $\mathbb{P}$ as having three main parts. The first part corresponds to the choice of ultrafilters in even coordinates — the $\mathcal{U}$'s of $r = (\mathcal{U}, r_1)$ for iterand conditions r; we call this the *Ultrafilter Part*. The next part corresponds to the coordinates where we have in addition non-trivial Mathias conditions (coordinates in SSup), we call it the *Mathias part*. Finally the odd coordinates, where the forcing chooses conditions in an arbitrary κ-centered, κ-directed closed forcing (coordinates in RSup), we call the *Directed Part*.

Extend $p_0 = p$ to a condition p_1 deciding whether $\dot{X}_0 \in \dot{\mathcal{U}}$, and let p_1' be the condition extending p_0 with the same ultrafilter part as p_1 and no other change from p_0. Then extend p_1' again to a condition p_2 which also makes a decision about $\dot{X}_0$ but either its Mathias or directed parts are incompatible with the ones corresponding to p_1; and correspondingly extend p_1' on its ultrafilter part to p_2'.

Continue extending the ultrafilter part, deciding whether or not $\dot{X}_0 \in \dot{U}$ with an antichain of different Mathias and directed parts until a maximal antichain is reached. This will happen in less than κ^+-many steps. If the resulting condition is called q_1 and has support $\alpha_1 < \Gamma^+$, then the set of conditions in $\mathbb{P}_{\alpha_1} \downarrow q_1$ which decide whether or not $\dot{X}_0$ belongs to $\dot{\mathcal{U}}$ is predense in $\mathbb{P}_{\alpha_1} \downarrow q_1$.

Repeat this process Γ-many times for each element in $\bar{X}$ until reaching a condition q_2 with the same property for all such names. Then do it for all $\mathbb{P}_{\alpha_1} \downarrow q_2$ names for subsets

[2]This is possible because κ is still supercompact in $V^{\mathbb{P}}$.

of κ and so on. Let q be the condition obtained once this overall process closes off with a fixed point. It follows, that if G is $\mathbb{P}$ generic containing q then $\dot{\mathcal{U}}^G \cap V[G_\alpha]$ is determined by G_α and therefore it is a normal ultrafilter U_α on κ in $V[G_\alpha]$. Now extend q once more to length $\alpha+1$ by choosing $\dot{\mathcal{U}}_\alpha$ to be the name for $\mathcal{U}_\alpha = \dot{U}^G \cap V[G_\alpha]$.

This argument gives us the desired property for a single $\alpha < \Gamma$. To have it for all $\alpha \in S \cup \{\sup S\}$ for an S of order type κ^*, we just have to iterate the process κ^*-many times (this is possible because $\kappa^* \leq \Gamma$), and then by cofinality considerations we see that moreover $\dot{\mathcal{U}} \cap V[G_{\sup S}] \in V[G_{\sup S}]$. $\square$

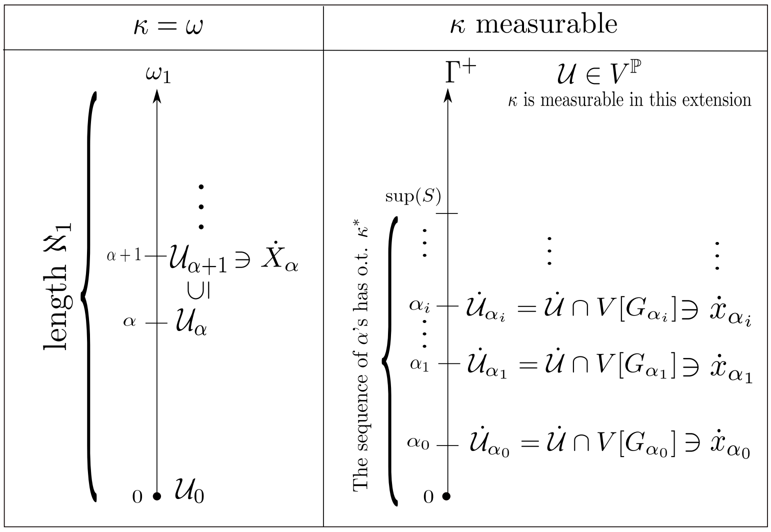

Figure 2.3.: Methods to find an ultrafilter with a small base

Remark 2.18. Note that, without loss of generality we can choose the domains of our conditions such that they have size Γ.

Working in the Laver-prepared model $V[G_{S_\kappa}]$, take S to be a set with the properties of the lemma above; this set will be fixed for the rest of the paper.

Now, using our Laver preparation S_κ and Laver diamond h we choose a supercompactness embedding $j : V \to M$ with critical point κ satisfying $j(\kappa) \geq \lambda$ where $\lambda \geq |S_\kappa * \mathbb{P}|$, $M^\lambda \subseteq M$ and $j(h)(\kappa) = (\mathbb{P}, \lambda)$. Then $j(S_\kappa) = S_\kappa * \dot{\mathbb{P}} * \dot{S}^*$ for an appropriate tail iteration $\dot{S}^*$ in M. Also if we denote $\mathbb{P}' = j(\mathbb{P})$ applying j to $S_\kappa * \dot{\mathbb{P}}$ we get $j(S_\kappa * \dot{\mathbb{P}}) = S_\kappa * \mathbb{P} * \dot{S}^* * (\mathbb{P}')^M$.

Consider then $j_0 : V[G_{S_\kappa}] \to M[G_{S_\kappa}][G_{\mathbb{P}}][H]$ where $G_{S_\kappa} * G_{\mathbb{P}} * H$ is generic for $j(S_\kappa)$. We want to lift again to $j^* : V[G_{S_\kappa}][G_{\mathbb{P}}] \to M[G_{S_\kappa}][G_{\mathbb{P}}][H][G_{\mathbb{P}'}]$ where $\mathbb{P}' = j_0(\mathbb{P})$.

We will do this by listing the maximal antichains below some master condition in $\mathbb{P}'$ extending every condition of the form $j_0(p)$ for $p \in G_{\mathbb{P}}$. The obvious master condition comes from choosing a lower bound p_0^* of $j_0''G_{\mathbb{P}}$.[3]

This condition has support contained in $j''\Gamma^+$ and for each $i < \Gamma^+$ even chooses the filter name $\dot{\mathcal{U}}_{j(i)}$ to be $j_0(\dot{\mathcal{U}}_i)$ as well as a $j(\kappa)$-Mathias condition with first component $\check{x}_i$, the Mathias generic added by $G_{\mathbb{P}}$ at stage i of the iteration. However we will choose a stronger master condition p^* with support contained in $j''\Gamma^+$ as follows:

1. If $i < \Gamma^+$ is an even ordinal and for each $A \in \mathcal{U}_i$ there is a $G_{\mathbb{P}_i}$-name $\dot{X}$ such that $A = X^{G_{\mathbb{P}_i}}$ and a condition $p \in G_{\mathbb{P}_i}$ such that $j_0(p) \Vdash \kappa \in j_0(\dot{X})$, then $p^*(j(i))$ is obtained from $p_0^*(j(i))$ by replacing the first component x_i of its $j(\kappa)$-Mathias condition by $x_i \cup \{\kappa\}$.
2. Otherwise $p^*(j(i)) = p_0^*(j(i))$.

Lemma 2.19. The condition p^* is an extension of p_0^*. If $G_{\mathbb{P}'}$ is chosen to contain p^*, j^* is the resulting lifting of j_0 and $\mathcal{U}$ is the resulting normal ultrafilter on κ derived from j^*, then whenever $\mathcal{U}_i$ is contained in $\mathcal{U}$, we have that $x_i \in \mathcal{U}$.

Proof. To show the first claim, it is enough to show that for all $i < \Gamma^+$ the condition p_i^* defined as p^* but replacing $x_{j(l)}$ by $x_{j(l)} \cup \{\kappa\}$ for $l < i$ satisfying (1.) extends p_0^*. We do this by induction on i. The base and limit cases are immediate. For the successor case, suppose we have the result for i and we want to prove it for $i+1$.

Let $G_{\mathbb{P}_j^*}(i)$ be any generic containing $p_i^* \upharpoonright j(i)$ and extend it to a generic $G_{\mathbb{P}^*}$ containing p_i^*. Hence, using the induction hypothesis $G_{\mathbb{P}^*}$ also contains p_0^* and therefore gives us a lifting j^* of j_0.

Now, any $p \in G_{\mathbb{P}}$ can be extended (inside $G_{\mathbb{P}}$) so that the Mathias condition it specifies at stage i is of the form $(s, A) \in \mathbb{M}^{\kappa}_{\mathcal{U}_i}$ where $s \subseteq x_i$ and $A \in \mathcal{U}_i$. Then using $(*)$ we infer $A = X^{G_{\mathbb{P}_i}}$ where $j_0(q) \Vdash \check{\kappa} \in j_0(\dot{X})$ for some $q \in G_{\mathbb{P}_i}$.

But then, since $p_0^* \in G_{\mathbb{P}^*}$, $j_0(q)$ is an element of $G_{\mathbb{P}_j^*(i)}$ and therefore:

$$\kappa \in j_0(\dot{X})^{G_{\mathbb{P}^*_{j(i)}}} = j^*(A).$$

It follows that the $j(\kappa)$-Mathias condition specified by $p_{i+1}^*(j(i))^{G_{\mathbb{P}^*_{j(i)}}}$ with first component $x_i \cup \{\kappa\}$ does extend the condition:

$$(x_i, j^*(A)) = (x_i, j_0(\dot{X})^{G_{\mathbb{P}^*_{j(i)}}}) \leq (s, j_0(\dot{X})^{G_{\mathbb{P}^*_{j(i)}}}).$$

This means that $p_i^* \upharpoonright j(i) \Vdash p_{i+1}^*(j(i)) \leq (s, j_0(\dot{X})) = j_0(p)(j(i))$ and thus p_{i+1}^* extends $j_0^*(p)$ for each $p \in G_{\mathbb{P}}$ and then also extends p_0^*.

To see the second claim, note that if $\mathcal{U}_i \subseteq \mathcal{U}$, then $\kappa \in j^*(A)$ for all $A \in \mathcal{U}_i$ which implies that $(*)$ is satisfied at i. Then $\kappa \in j^*(x_i)$ and so $x_i \in \mathcal{U}$. $\square$

[3] This condition exists because $j_0''G_{\mathbb{P}}$ is directed and the forcing is sufficiently directed-closed.

The following is the main result of the section, it shows that there is a generic extension where the ultrafilter number on κ is consistently in the interval $(\kappa^+, 2^\kappa)$.

Theorem 2.20. *Suppose κ is a supercompact cardinal and κ^* is a regular cardinal with $\kappa < \kappa^* \leq \Gamma$, $\Gamma^\kappa = \Gamma$. There is a forcing notion $\mathbb{P}^*$ preserving cofinalities such that $V^{\mathbb{P}^*} \models \mathfrak{u}(\kappa) = \kappa^* \wedge 2^\kappa = \Gamma$.*

Proof. We will not work with the whole generic extension given by $\mathbb{P}$. In fact we will chop the iteration at the step $\alpha = \sup(S)$ (as in the Lemma 2.17) which is an ordinal of cofinality κ^*. Define $\mathbb{P}^* = \mathbb{P}_\alpha$.

Take G to be a $\mathbb{P}^*$-generic filter. The fact that $2^\kappa = \Gamma$ in the extension is a consequence of the fact that the domains of the conditions obtained in Lemma 2.17 can be chosen in such a way that they all have size Γ.

To prove $\mathfrak{u}(\kappa) = \kappa^*$ we consider the ultrafilter $\mathcal{U}^*$ on κ given by the restriction of $\mathcal{U}$ (Lemma 2.17). Then by the same lemma note that for all $i \in S$ the restriction of $\mathcal{U}$ to the model $V[G_i]$ belongs to $V[G_{i+1}]$ and moreover, this is the ultrafilter $\mathcal{U}_i^G$ chosen generically at stage i.

Furthermore by our choice of Master Conditions the κ-Mathias generics $\dot{x}_i$ belong to $\mathcal{U}$. Then $\mathcal{U}^*$ is generated by $\dot{x}_i$ for $i \in S$. The other inequality $\mathfrak{u}(\kappa) \geq \kappa^*$ is a consequence of $\mathfrak{b}(\kappa) \geq \kappa^*$ (see discussion below) and the following simple Proposition. □

Proposition 2.21. $\mathfrak{b}(\kappa) \leq \mathfrak{r}(\kappa)$ and $\mathfrak{r}(\kappa) \leq \mathfrak{u}(\kappa)$.

Proof. The first is the consequence of the following property that can be directly generalized from the countable case: there are functions $\Phi : [\kappa]^\kappa \to \kappa^{\uparrow\kappa}$ and $\Psi : \kappa^{\uparrow\kappa} \to [\kappa]^\kappa$ such that whenever $\Phi(A) \leq^* f$ then $\Psi(f)$ splits A.

For the second inequality, it is just necessary to notice that if $\mathcal{B}$ is a base for a uniform ultrafilter on κ, then $\mathcal{B}$ cannot be split by a single set X. Otherwise neither X nor $\kappa \setminus X$ will belong to the ultrafilter. □

Discussion: In Chapter 1, one of the models that we studied was the generic extension obtained after a $< \kappa$-support iteration of κ-Mathias forcing respect to some ultrafilters chosen at every stage of the iteration (see Section 1.2.4). Moreover we proved the following property:

Lemma (Lemma 1.61)**.** If $\mathcal{U}$ is a normal ultrafilter on κ, then $\mathbb{M}^\kappa_{\mathcal{U}}$ and $\mathbb{L}^\kappa_{\mathcal{U}}$ are forcing equivalent.

And as a consequence we obtained:

Corollary (Corollary 1.62)**.** If $\mathcal{U}$ is a normal ultrafilter on κ then $\mathbb{M}^\kappa_{\mathcal{U}}$ always adds dominating functions.

Note, that in the even steps that belong to the ultrafilter support we are basically using κ-Mathias forcing respect to some ultrafilter that has been chosen generically. Nevertheless, putting this results together, we conclude that, at this steps of the iteration we are adding κ-dominating functions that clearly witness $\mathfrak{b}(\kappa) \geq \kappa^*$.

2.3. Other generalized cardinal invariants in the model

If we review the proof of Theorem 2.20, it has to be pointed out that the odd steps of our iteration (i.e. the steps where we used a name for a κ-directed closed forcing notion) were not used at all. The reason to introduce them is to force the values of other cardinal invariants to be decided in our model. This section shows first the posets we will use in the odd steps and the cardinal invariants associated to them.

2.3.1. κ-maximal almost disjoint families

We start with the generalization of the almost disjointness number for uncountable cardinals, which was introduced by Blass-Hyttinen and Zhang in [BHZ].

Definition 2.22. Two sets A and $B \in \mathcal{P}(\kappa)$ are called *κ-almost disjoint* if $A \cap B$ has size $<\kappa$. We say that a family of sets $\mathcal{A} \subseteq \mathcal{P}(\kappa)$ is *κ-almost disjoint* if it has size at least κ and all its elements are pairwise κ-almost disjoint. A family $\mathcal{A} \subseteq [\kappa]^\kappa$ is called a *κ-maximal almost disjoint* (abbreviated κ-mad) if it is κ-almost disjoint and is not properly included in another such family.

Definition 2.23. $\mathfrak{a}(\kappa) = \min\{|\mathfrak{A}|\colon \mathfrak{A}$ is a κ-mad family$\}$

Proposition 2.24. $\mathfrak{b}(\kappa) \leq \mathfrak{a}(\kappa)$

Proof. Suppose $\mathfrak{a}(\kappa) = \lambda$, let $\mathfrak{A} = \{A_\alpha : \alpha < \lambda\}$ be a κ-almost disjoint family where $\lambda < \mathfrak{b}(\kappa)$. For each $\alpha < \kappa$, let $\tilde{A}_\alpha = A_\alpha \setminus \bigcup_{\delta<\alpha}(A_\alpha \cap A_\delta)$. Since $\mathfrak{A}$ is κ-ad, we have $|\tilde{A}_\alpha| = \kappa$, also $\tilde{A}_\alpha \cap \tilde{A}_\beta = \emptyset$ for all $\alpha, \beta < \kappa$. Thus, $\tilde{A}_\alpha =^* A_\alpha$. (Here $*$ means modulo a set of size $<\kappa$).

Whenever $g \in \kappa^\kappa$, define $e^\alpha_g = \text{next}(\tilde{A}_\alpha, g(\alpha))$, the least ordinal in $\tilde{A}_\alpha$ greater than $g(\alpha)$. Let $E_g = \{e^\alpha_g : \alpha < \kappa\}$. Then E_g contains one element of each $\tilde{A}_\alpha$, so it is unbounded in κ. Also $|E_g \cap \tilde{A}_\alpha| < \kappa$, for all $\alpha < \kappa$.

Now when $\kappa \leq \alpha < \lambda$. Each $A_\alpha \cap A_\gamma$, has size less than κ, so we can fix f_α such that for all $\gamma < \kappa$ all elements of $A_\alpha \cap A_\gamma$ are less than $f_\alpha(\gamma)$. Where $f_\alpha(\gamma) = \sup(A_\alpha \cap A_\gamma) + 1$.

Now consider $\{f_\alpha : \alpha < \lambda\}$, which is a family of $\lambda < \mathfrak{b}(\kappa)$ functions, therefore there exists $g \in \kappa^\kappa$ with the property $f_\alpha <^* g$, for all α.

As consequence we have that $E_g \cap A_\alpha$ has size less than κ, for all α because if $e^\gamma_g \in E_g \cap A_\alpha$ then $e^\gamma_g \in \tilde{A}_\alpha$ and $e^\gamma_g > g(\alpha)$, so $f_\alpha(\gamma) > e^\gamma_g > g(\gamma)$ which is only possible for a set of less than κ values.

Therefore, $\mathfrak{A}$ is not maximal. Then $\mathfrak{b}(\kappa) \leq \lambda$. $\square$

The following forcing notion is a generalized version of Hechler's poset to add a mad family of subsets of ω (see for example [BF11]).

Definition 2.25. Let $\mathcal{A} = \{A_i\}_{i<\delta}$ be a κ-almost disjoint family. Let $\bar{\mathbb{Q}}(\mathcal{A}, \kappa)$ be the poset of all pairs (s, F) where $s \in 2^{<\kappa}$ and $F \in [\mathcal{A}]^{<\kappa}$, with extension relation stating that $(t, H) \leq (s, F)$ if and only if $t \supseteq s$, $H \supseteq F$ and for all $i \in \mathrm{dom}(t) \setminus \mathrm{dom}(s)$ with $t(i) = 1$ we have $i \notin \bigcup\{A : A \in F\}$

Note that the poset $\bar{\mathbb{Q}}(\mathcal{A}, \kappa)$ is κ-centered and κ-directed closed. If G is $\bar{\mathbb{Q}}(\mathcal{A}, \kappa)$-generic then $\chi_G = \bigcup\{t : \exists F(t, F) \in G\}$ is the characteristic function of an unbounded subset x_G of κ such that $\forall A \in \mathcal{A}(|A \cap x_G|) < \kappa$.

Proposition 2.26. If $Y \in [\kappa]^\kappa \setminus \mathcal{I}_\mathcal{A}$, where $\mathcal{I}_\mathcal{A}$ is the κ-complete ideal generated by the κ almost disjoint family $\mathcal{A}$, then $\Vdash_{\bar{\mathbb{Q}}(\mathcal{A},\kappa)} |Y \cap \dot{x}_G| = \kappa$.

Proof. Let $(s, F) \in \bar{\mathbb{Q}}(\mathcal{A}, \kappa)$ and $\alpha < \kappa$ be arbitrary. It is sufficient to show that there are $(t, H) \leq (s, F)$ and $\beta > \alpha$ such that $(t, H) \Vdash \beta \in \check{Y} \cap \dot{x}_G$. Since $\kappa \setminus \bigcup F$ is unbounded and $Y \notin \mathcal{I}_\mathcal{A}$, we have that $|Y \setminus \bigcup F| = \kappa$. Take any $\beta > \alpha$ in $Y \setminus \bigcup F$ and define $t' = t \cup \{(\beta, 1)\} \cup \{(\gamma, 0) : \sup(\mathrm{dom}(t)) < \gamma < \beta\}$. Then (t', H) is as desired. □

2.3.2. The generalized splitting, reaping and independence numbers

The generalized splitting number was first studied by Suzuki in [Suz98].

Definition 2.27. For A and $B \in \wp(\kappa)$, say $A \subseteq^* B$ (A is almost contained in B) if $A \setminus B$ has size $<\kappa$. We also say that A *splits* B if both $A \cap B$ and $B \setminus A$ have size κ. A family $\mathcal{A}$ is called a *splitting family* if every unbounded (with supremum κ) subset of κ is split by a member of $\mathcal{A}$. Finally $\mathcal{A}$ is *unsplit* if no single set splits all members of $\mathcal{A}$.

Definition 2.28.

- $\mathfrak{s}(\kappa) = \min\{|\mathcal{A}|: \mathcal{A}$ is a splitting family of subsets of $\kappa\}$.
- $\mathfrak{r}(\kappa) = \min\{|\mathcal{A}|: \mathcal{A}$ is an unsplit family of subsets of $\kappa\}$.

Definition 2.29 (The generalized independence number)**.** A family $\mathcal{I} = \{I_\delta : \delta < \mu\}$ of subsets of κ is called κ-independent if for all disjoint $I_0, I_1 \subseteq \mathcal{I}$, both of size $<\kappa$, $\bigcap_{\delta \in I_0} I_\delta \cap \bigcap_{\delta \in I_0} (I_\delta)^c$ is unbounded in κ. The generalized independence number $\mathfrak{i}(\kappa)$ is defined as follows:

$$\mathfrak{i}(\kappa) = \min\{|\mathcal{I}|: \mathcal{I} \text{ is an independent family of subsets of } \kappa\}.$$

Even though we do not know the exact value of $\mathfrak{i}(\kappa)$ in the model we construct (see Theorem 2.43), the inequality which we prove below and the value we fix for $\mathfrak{d}(\kappa)$ in this model, will provide a constraint for $\mathfrak{i}(\kappa)$.

Proposition 2.30. If $\mathfrak{d}(\kappa)$ is such that for every $\gamma < \mathfrak{d}(\kappa)$ we have $\gamma^{<\kappa} < \mathfrak{d}(\kappa)$, then $\mathfrak{d}(\kappa) \leq \mathfrak{i}(\kappa)$

The proof will be essentially a modification of the one for the countable case (Theorem 5.3 in [Bla10]). To obtain the above proposition, we will need the following lemma.

Lemma 2.31. Suppose $\mathcal{C} = (C_\alpha : \alpha < \kappa)$ is a $\subseteq^*$-decreasing sequence of unbounded subsets of κ and $\mathcal{A}$ is a family of less than $\mathfrak{d}(\kappa)$ many subsets of κ such that each set in $\mathcal{A}$ intersects every C_α in a set of size κ. Then $\mathcal{C}$ has a pseudointersection B that also has unbounded intersection with each member of $\mathcal{A}$.

Proof. Without loss of generality assume that the sequence $\mathcal{C}$ is $\subseteq$-decreasing. For any $h \in \kappa^\kappa$ define $B_h = \bigcup_{\alpha<\kappa}(C_\alpha \cap h(\alpha))$, clearly B_h is a pseudointersection of $\mathcal{C}$. Thus, we must find $h \in \kappa^\kappa$ such that $|B_h \cap A| = \kappa$ for each $A \in \mathcal{A}$.

For each $A \in \mathcal{A}$ define the function $f_A \in \kappa^\kappa$ as follows: $f_A(\beta) =$ the β-th element of $A \cap C_\beta$. The set $\{f_A : A \in \mathcal{A}\}$ has cardinality $< \mathfrak{d}(\kappa)$, then we can find $h \in \kappa^\kappa$ such that for all $A \in \mathcal{A}$, $h \not\leq^* f_A$ (i.e. $X_A = \{\delta < \kappa : f_A(\delta) < h(\delta)\}$ is unbounded).

Then B_h will be the pseudointersection we need. Note that $B_h \cap A = \bigcup_{\alpha<\kappa}(C_\alpha \cap A) \cap h(\alpha) \supseteq \bigcup_{\alpha \in X_A}(C_\alpha \cap A) \cap f_A(\alpha)$ which is unbounded. □

Proof of Proposition 2.30. Suppose that $\mathcal{I}$ is an independent family of cardinality $< \mathfrak{d}(\kappa)$, we will show it is not maximal. For this purpose choose $\mathcal{D} = (D_\alpha : \alpha < \kappa) \subseteq \mathcal{I}$ and let $\mathcal{I}' = \mathcal{I} \setminus \mathcal{D}$.

For each $f : \kappa \to 2$ consider the set $C_\alpha = \bigcap_{\beta<\alpha} D_\beta^{f(\beta)}$ where $D^0 = D$ and $D^1 = D^c$, also define $\mathcal{A} = \{\bigcap I_0 \setminus \bigcup I_1 : I_0 \text{ and } I_1 \text{ are disjoint subfamilies of } \mathcal{I} \text{ of size } < \kappa\}$. Since $|\mathcal{I}|^{<\kappa} < \mathfrak{d}(\kappa)$, the family $\mathcal{A}$ has size $< \mathfrak{d}(\kappa)$.

Then, using the lemma before there exists a pseudointersection B_f of the family $(C_\alpha : \alpha < \kappa)$ that intersects in an unbounded set all members of $\mathcal{A}$. Then if $f \neq g$ we have $|B_f \cap B_g| < \kappa$ (Moreover, we can suppose they are disjoint).

Now, fix two disjoint dense subsets X and X' of 2^κ. Take $Y = \bigcup_{f \in X} B_f$ and $Y' = \bigcup_{f \in X'} B_f$, note that $Y \cap Y' = \emptyset$. Then it is enough to show that both Y and Y' have intersection of size κ with each member of $\mathcal{A}$. We write the argument for Y (for Y' i is analogous).

Take $J_0, J_1 \subseteq \mathcal{I}$ both of size $< \kappa$, call J_0', J_1' their intersections with $\mathcal{I}'$. There exists $\alpha < \kappa$ such that if D_β belongs to J_0 or J_1, then $\beta < \alpha$ and using the density of the sets X fix $f \in X$ such that, if $D_\beta \in J_0 \cup J_1$, then $f(\beta) = 0$ or 1 respectively. Hence:

$$\begin{aligned} \bigcap J_0 \setminus \bigcup J_1 &= \bigcap J_0' \setminus \bigcup J_1' \cap \bigcap_{\{\beta : D_\beta \in J_0 \cup J_1\}} D_\beta^{f(\beta)} \\ &\supseteq \bigcap J_0' \setminus \bigcup J_1' \cap \bigcap_{\beta<\alpha} D_\beta^{f(\beta)} \\ &{}^* \supseteq \bigcap J_0' \setminus \bigcup J_1' \cap B_f \text{ which is unbounded.} \end{aligned} \tag{2.1}$$

□

2.3.3. The generalized pseudointersection and tower numbers

The generalizations of these two cardinals to the uncountable introduced by Garti in [Gar11].

Definition 2.32. Let $\mathcal{F}$ be a family of subsets of κ, we say that $\mathcal{F}$ has the *strong intersection property* (SIP) if any subfamily $\mathcal{F}' \subseteq \mathcal{F}$ of size $< \kappa$ has intersection of size κ, we also say that $A \subseteq \kappa$ is a *pseudointersection* of $\mathcal{F}$ is $A \subseteq^* F$, for all $F \in \mathcal{F}$. A *tower* $\mathcal{T}$ is a well-ordered family of subsets of κ with the SIP that has no pseudointersection of size κ.

- The generalized pseudointersection number $\mathfrak{p}(\kappa)$ is defined as the minimal size of a family $\mathcal{F}$ which has the SIP but no pseudointersection of size κ.
- The generalized tower number $\mathfrak{t}(\kappa)$ is defined as the minimal size of a tower $\mathcal{T}$ of subsets of κ.

Lemma 2.33. $\kappa^+ \leq \mathfrak{p}(\kappa) \leq \mathfrak{t}(\kappa) \leq \mathfrak{b}(\kappa)$

Proof. First we prove $\kappa^+ \leq \mathfrak{p}(\kappa)$: Take a family of subsets of κ, $\mathcal{B} = (B_\alpha : \alpha < \kappa)$ with the SIP. Then we can construct a new family $\mathcal{B}' = (B'_\alpha : \alpha < \kappa)$ such that $B'_{\alpha+1} \subseteq B'_\alpha$ and $B'_\alpha \subseteq B_\alpha$ for all $\alpha < \kappa$. Simply define $B'_0 = B_0$, $B'_{\alpha+1} = B_{\alpha+1} \cap B'_\alpha$ and for limit γ, $B'_\gamma = \bigcap_{\alpha<\gamma} B'_\alpha$. Note that this construction is possible thanks to the SIP.

Then, without loss of generality we can find κ-many indexes β where it is possible to choose $a_\beta \in B'_\alpha \setminus B'_{\alpha+1}$. Hence the set $X = \{a_\beta : \beta < \kappa\}$ is a pseudointersection of the family $\mathcal{B}'$ and so of $\mathcal{B}$.

$\mathfrak{p}(\kappa) \leq \mathfrak{t}(\kappa)$ is immediate from the definitions and $\mathfrak{t}(\kappa) \leq \mathfrak{b}(\kappa)$ was proven in [Gar11, Claim 1.8]. □

2.3.4. The generalized distributivity number

In order generalize the distributivity number we have to consider a modification of it. In the countable case $\mathfrak{h}$ is defined as the minimum cardinal λ such that the poset $\mathcal{P}(\omega)/\operatorname{fin}$ is not λ-distributive, where a poset $\mathbb{P}$ is λ-distributive if every collection $\mathcal{D}$ of λ-many dense open sets has dense open intersection.

If we consider the poset $\mathcal{P}(\kappa)/ < \kappa$, then it is not clear that the minimum cardinal λ such that it is not λ-distributive is greater even than ω (see discussion at the end of the section). Hence we will start from a non-principal κ-complete ultrafilter $\mathcal{W}$, and consider it as a poset ordered by inclusion modulo the ideal of sets of size less than κ.

Definition 2.34. The Generalized Distributivity Number with respect to the κ-complete ultrafilter $\mathcal{W}$, $\mathfrak{h}_{\mathcal{W}}(\kappa)$ is defined as the minimal $\lambda \geq \kappa$ for which $\mathcal{W}$ is not λ-distributive.

Proposition 2.35. $\mathfrak{p}(\kappa) \leq \mathfrak{h}_{\mathcal{W}}(\kappa) \leq \mathfrak{s}(\kappa)$.

Proof.

• $\mathfrak{p}(\kappa) \leq \mathfrak{h}_{\mathcal{W}}(\kappa)$: Let $\lambda < \mathfrak{p}(\kappa)$. We shall prove that $\mathcal{F}$ is λ-distributive. For this purpose take $(D_\alpha : \alpha < \lambda)$ to be a family of open dense sets in $\mathcal{W}$.

Using density as well as the completeness of the filter $\mathcal{W}$ it is possible to construct a sequence $\bar{X} = (X_\alpha : \alpha < \lambda)$ such that, for all $\alpha < \lambda$, $X_\alpha \in D_\alpha$. By induction start with $X_0 \in D_0$ and given X_α, take $X_{\alpha+1} \in D_{\alpha+1}$ where $X_{\alpha+1} \leq X_\alpha$. For the limit step $\alpha < \lambda$, take $Y = \bigcap_{\beta<\alpha} X_\beta \in \mathcal{W}$ and then $X_\alpha \in D_\alpha$, $X_\alpha \leq Y$.

Thus the family $\bar{X}$ has the SIP and since $\lambda < \mathfrak{p}(\kappa)$ has a pseudointersection $X \in \mathcal{W}$ which belongs to $\bigcap_{\alpha<\lambda} D_\alpha$.

• $\mathfrak{h}_{\mathcal{F}}(\kappa) \leq \mathfrak{s}(\kappa)$: Let $\mathcal{S}$ be a splitting family of subsets of κ. For each $S \in \mathcal{S}$, the set $D_S = \{X \in \mathcal{W} : X \text{ is not split by } S\}$ is dense open. Because $\mathcal{S}$ is a splitting family we obtain $\bigcap_{S\in\mathcal{S}} D_S = \emptyset$. □

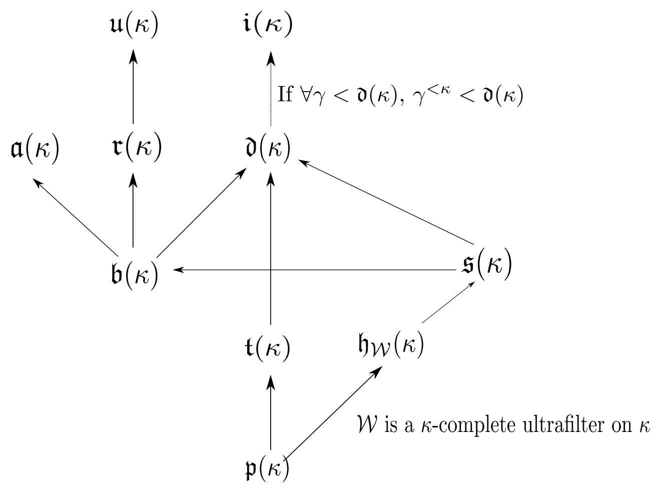

Figure 2.4.: Figure 4: Provable inequalities for κ-measurable.

A note on the definitions of the tower, pseudointersection and distributivity numbers

The definitions of the generalized versions of this cardinals are not simply the straightforward ones, the reason why we work with the definitions presented in the section above is because among others, our objective is that at least these cardinals are somehow “typical ”, in the sense that at least we would like them to have values in the interval $[\kappa^+, 2^\kappa]$. Unpublished work from of Will Brian and Jonathan Verner shows for instance that the elementary generalization of the pseudointersection and tower numbers to κ has value ω. Namely if we define:

Definition 2.36 (Other approximations to the pseudointersection and tower numbers)**.**

- $\mathfrak{p}^*(\kappa) = \min\{|\mathcal{F}|: \mathcal{F}$ is a family with the FIP and no pseudointersection of size $\kappa\}$.
- $\mathfrak{t}^*(\kappa) = \min\{|\mathcal{T}|: \mathcal{T}$ is a well-ordered family of subsets of κ without pseudointersection of size $\kappa\}$.

Here FIP means that for any finite sub-family $\mathcal{F}' \subseteq \mathcal{F}$, $\bigcap \mathcal{F}'$ has size κ.

Proposition 2.37 (Brian-Verner)**.** If κ is a cardinal with uncountable cofinality, then $\mathfrak{p}^*(\kappa) = \mathfrak{t}^*(\kappa) = \aleph_0$.

Proof. Clearly we still have the inequality $\mathfrak{p}^*(\kappa) \leq \mathfrak{t}^*(\kappa)$. Let $\kappa = \bigcup n \in \omega X_n$ be a partition of κ into sets of size κ and define $T_n = \bigcup_{m>n} X_m$. Note that every set X with $|X|= \kappa$, since $\bigcap_{n\in\omega} T_n = \emptyset$ for each $\alpha < \kappa$ there exists $n_\alpha \in \omega$ such that $\alpha \notin T_{n_\alpha}$. Finally using that κ has uncountable cofinality we can find $n \in \omega$ such that the set $Y = \{\alpha < \kappa : n_\alpha = n\}$ is unbounded on κ and certainly $Y \subseteq X \setminus T_n$ which implies that the family of sets $(T_n : n \in \omega)$ cannot have a pseudointersection of size κ. □

Moreover, they obtained the following in the case when κ is an uncountable singular cardinal.

Proposition 2.38 (B-V)**.** If κ is an uncountable cardinal with $\mathrm{cf}(\kappa) = \omega$, then $\mathfrak{t}^*(\kappa)$ is uncountable.

Proof. Write $\kappa = \sup_{n\in\omega} \kappa_n$ be a cofinal sequence of regular cardinals on κ and let $\mathcal{T} = \{T_n : n \in \omega\}$ be a descending sequence of unbounded sets of κ. Recursively construct a sequence $(a_n : n \in \omega)$ of natural numbers such that $|T_i \cap [\kappa_{a_i}, \kappa_{a_{i+1}})|\leq |T_n \cap [\kappa_{a_n}, \kappa_{a_{n+1}})|$ for each $i < n$. Putting $T = \bigcup_{n\in\omega} T_n \cap [\kappa_{a_n}, \kappa_{a_{n+1}})$, we obtain $|T|= \kappa$ and T is a pseudointersection of $\mathcal{T}$. □

Proposition 2.39 (B-V)**.** If κ is an uncountable cardinal with $\mathrm{cf}(\kappa) = \omega$, then $\mathfrak{p}^* = \omega_1$.

Proof. Let $\kappa = \sup_{n\in\omega} \kappa_n$ be a cofinal sequence of regular cardinals on κ, it is enough to find a well-ordered family of size ω_1 without pseudointersection of size κ. For each n fix a sequence of sets $\mathcal{F}_n = (F^n_\alpha : \alpha < \omega) \subseteq \mathcal{P}([\kappa_n, \kappa_{n+1}))/ < \kappa_{n+1}$ and such that every finite subfamily of $\mathcal{F}_n$ has a unbounded intersection yet every infinite subfamily $\mathcal{G} \subseteq \mathcal{F}_n$ has empty intersection.

Put $\mathcal{F}_\alpha = \bigcup_{n\in\omega} F^n_\alpha$, we claim that there is no pseudointersection of $\mathcal{F}$, if so let X to be such a set and define $Y = X \setminus \mathcal{F}_\alpha$ which by assumption is a set of size $<\kappa$, so there is an n_α such that $|Y_\alpha|< \kappa_n$ and by pigeonhole principle there is an unbounded set $A \subseteq \omega_1$ and a single $n \in \omega$ such that $|Y_\alpha|< \kappa_n$, for all $\alpha \in A$.

Let $Y = \bigcup_{\alpha \in A} Y_\alpha$ and take $Z = X \setminus Y$, then Z is unbounded and we can choose $m > n$ such that $Z \cap [\kappa_m, \kappa_{m+1}) \neq \emptyset$. However $Z \subseteq \mathcal{F}_\alpha$ for each $\alpha \in A$, so in particular $Z \cap [\kappa_m, \kappa_{m+1}) \subseteq \mathcal{F}_\alpha \cap [\kappa_m, \kappa_{m+1}) = F^m_\alpha$ which is a contradiction. $\square$

What about the distributivity number? In this case, the motivation to work with the definitions we gave comes from the following results and again our choice is justified because we prefer to work with cardinal invariants on κ which are $\geq \kappa^+$. The following results are due to Balcar and Vopěnka Suppose we work with the algebra $\mathcal{P}(\kappa)/ < \kappa$, when κ is an uncountable cardinal and we define $\mathfrak{h}(\kappa)$ as we do in the countable case, namely:

Definition 2.40. $\mathfrak{h}(\kappa) = \min\{\lambda : \text{the poset } \mathcal{P}(\kappa)/ < \kappa \text{ is not } \lambda\text{- distributive}\}$

Theorem 2.41 (Theorem 2.5 in [BV72])**.** *Let κ be an uncountable cardinal. If* $\mathrm{cf}(\kappa) = \omega$*, then* $\mathfrak{h}(\kappa) = \omega$*, otherwise* $\mathfrak{h}(\kappa) = \omega_1$.

The proofs are quite similar to the ones for the pseudointersection and tower numbers, so we omit them.

2.4. Applications

This final section shows some generic extensions constructed from our main model can help to decide other cardinal invariants, including the defined in the section above and some of the ones in the generalized Cichoń's diagram (see Chapter 1). We will stick to the notation on this chapter, let κ, κ^*, Γ, α and $\mathbb{P}^*$ be fixed as in Theorem 2.20.

Theorem 2.42. *Let G be $\mathbb{P}^*$-generic. Then $V[G]$ satisfies* $\mathrm{add}(\mathcal{M}_\kappa) = \mathrm{cof}(\mathcal{M}_\kappa) = \mathrm{non}(\mathcal{M}_\kappa) = \mathrm{cov}(\mathcal{M}_\kappa) = \mathfrak{s}(\kappa) = \mathfrak{r}(\kappa) = \mathfrak{d}(\kappa) = \mathfrak{b}(\kappa) = \kappa^*$.

Proof. Note that $\mathfrak{b}(\kappa) \geq \kappa^*$ because any set of functions in κ^κ of size $< \kappa^*$ appears in some initial part of the iteration (by Lemma 2.15) and so is dominated by the Mathias generic functions added at later stages. On the other hand, any cofinal sequence of length κ^* of the Mathias generics forms a dominating family. Thus $\mathfrak{d}(\kappa) \leq \kappa^*$ and since clearly $\mathfrak{b}(\kappa) \leq \mathfrak{d}(\kappa)$, we obtain $V^{\mathbb{P}^*} \vDash \mathfrak{b}(\kappa) = \mathfrak{d}(\kappa) = \kappa^*$.

To see that $\mathfrak{s}(\kappa) \geq \kappa^*$, observe that a Mathias generic subset of κ is unsplit by any ground model subset of κ and that every family of κ-reals of size less than κ^* is contained in $V^{\mathbb{P}_\beta}$ for some $\beta < \alpha$. On the other hand, since $\mathbb{P}^*$ is locally a $(< \kappa)$-support iteration, it does add κ-Cohen reals. Any cofinal sequence of length κ^* of such κ-Cohen reals forms a splitting family and so $V^{\mathbb{P}^*} \vDash \mathfrak{s}(\kappa) \leq \kappa^*$. Thus $V^{\mathbb{P}^*} \vDash \mathfrak{s}(\kappa) = \kappa^*$. That $\mathfrak{r}(\kappa) = \kappa^*$ follows from Proposition 2.21.

To verify the values of the characteristics associated to $\mathcal{M}_\kappa$, proceed as follows. Since $\mathfrak{b}(\kappa) \leq \mathrm{non}(\mathcal{M}_\kappa)$, $V^{\mathbb{P}^*} \vDash \kappa^* \leq \mathrm{non}(\mathcal{M}_\kappa)$. On the other hand, any cofinal sequence of κ-Cohen reals of length κ^* is a non-meager set and so a witness to $\mathrm{non}(\mathcal{M}_\kappa) \leq \kappa^*$. By a similar argument and the fact that $\mathfrak{d}(\kappa) = \kappa^*$ in $V^{\mathbb{P}^*}$, we obtain that $V^{\mathbb{P}^*} \vDash \mathrm{cov}(\mathcal{M}_\kappa) = \kappa^*$. Now, Lemma 1.27 implies that $\mathrm{add}(\mathcal{M}_\kappa) = \kappa^* = \mathrm{cof}(\mathcal{M}_\kappa)$. $\square$

Now, we are ready to prove our main theorem.

Theorem 2.43. *Suppose κ is a supercompact cardinal, κ^* is a regular cardinal with $\kappa < \kappa^* \leq \Gamma$ and Γ satisfies $\Gamma^\kappa = \Gamma$. Then there is forcing extension in which cardinals have not been changed satisfying:*

$$\begin{aligned}\kappa^* &= \mathfrak{u}(\kappa) = \mathfrak{b}(\kappa) = \mathfrak{d}(\kappa) = \mathfrak{a}(\kappa) = \mathfrak{s}(\kappa) = \mathfrak{r}(\kappa) = \mathrm{cov}(\mathcal{M}_\kappa) \\ &= \mathrm{add}(\mathcal{M}_\kappa) = \mathrm{non}(\mathcal{M}_\kappa) = \mathrm{cof}(\mathcal{M}_\kappa) \text{ and } 2^\kappa = \Gamma.\end{aligned}$$

If in addition $(\Gamma)^{<\kappa^*} = \Gamma$ then we can also provide that $\mathfrak{p}(\kappa) = \mathfrak{t}(\kappa) = \mathfrak{h}_{\mathcal{W}}(\kappa) = \kappa^*$ where $\mathcal{W}$ is a κ-complete ultrafilter on κ.

Proof. We will modify the iteration $\mathbb{P}^*$ to an iteration $\bar{\mathbb{P}}^*$ by specifying the iterands $\dot{\mathbb{Q}}_j$ for every odd ordinal $j < \alpha$. Let $\bar{\gamma} = \langle \gamma_i \rangle_{i<\kappa^*}$ be a strictly increasing cofinal in α sequence of odd ordinals. The stages in $\bar{\gamma}$ will be used to add a κ-maximal almost disjoint family of size κ^*.

If $\Gamma^{<\kappa^*} = \Gamma$, then using an appropriate bookkeeping function F with domain the odd ordinals in α which are not in the cofinal sequence $\bar{\gamma}$ we can use the generalized Mathias poset to add pseudointersections to all filter bases of size $<\kappa^*$ with the SIP. In case $\Gamma^{<\kappa^*} > \Gamma$, just take for odd stages which are not in $\bar{\gamma}$ arbitrary κ-centered, κ-directed closed forcing notions of size at most Γ, such as the trivial forcing.

To complete the definition of $\bar{\mathbb{P}}^*$ it remains to specify the stages in $\bar{\gamma}$. Fix a ground model κ-ad family $\mathcal{A}_0$ of size κ and let $\mathbb{Q}_{\gamma_0} = \bar{\mathbb{Q}}(\mathcal{A}_0, \kappa)$ (see Definition 2.25). Now, fix any $i < \kappa^*$. For each $j < i$ let $\bar{x}_{\gamma_j}$ be the generic subset of κ added by $\mathbb{Q}_{\gamma_j} = \bar{\mathbb{Q}}(\mathcal{A}_j, \kappa)$ where $\mathcal{A}_j = \mathcal{A}_0 \cup \{\bar{x}_{\gamma_k}\}_{k<j}$.

With this the recursive definition of the iteration $\bar{\mathbb{P}}^*$ is defined. In $V^{\bar{\mathbb{P}}^*}$ let $\mathcal{A}_* = \mathcal{A}_0 \cup \{\bar{x}_{\gamma_j}\}_{j<\kappa^*}$. We will show that $\mathcal{A}_*$ is a κ-mad family. Clearly $\mathcal{A}_*$ is κ-ad. To show maximality of $\mathcal{A}_*$, consider an arbitrary $\bar{\mathbb{P}}^*$-name $\dot{X}$ for a subset of κ and suppose $\Vdash_{\bar{\mathbb{P}}^*} (\{\dot{X}\} \cup \mathcal{A}_* \text{ is } \kappa\text{-ad})$. By Corollary 2.15 $\dot{X}$ can be viewed as a $\bar{\mathbb{P}}^*_\beta$-name for some $\beta < \alpha$. Then for $\gamma_j > \beta$, by Proposition 2.26 we obtain $V^{\bar{\mathbb{P}}^*_\beta} \models |\bar{x}_{\gamma_j} \cap \dot{X}| = \kappa$, which is a contradiction. Thus $\mathcal{A}_*$ is indeed maximal and so $\mathfrak{a}(\kappa) \leq \kappa^*$. However in $V^{\bar{\mathbb{P}}^*}$, $\mathfrak{b}(\kappa) = \kappa^*$ and since $\mathfrak{b}(\kappa) \leq \mathfrak{a}(\kappa)$ (Proposition 2.26) we obtain $V^{\,b\bar{b}P^*} \vDash \mathfrak{a}(\kappa) = \kappa^*$.

Suppose $\Gamma^{<\kappa^*} \leq \Gamma$. In this case, every filter of size $<\kappa^*$ with the SIP has a pseudointersection in $V^{\bar{\mathbb{P}}^*}$. Thus in the final extension $\mathfrak{p}(\kappa) \geq \kappa^*$. However $\mathfrak{p}(\kappa) \leq \mathfrak{t}(\kappa) \leq \mathfrak{s}(\kappa)$ and since $V^{\bar{\mathbb{P}}^*} \vDash \mathfrak{s}(\kappa) = \kappa^*$, we obtain that $\mathfrak{p}(\kappa) = \mathfrak{t}(\kappa) = \kappa^*$. By Proposition 2.35, $\mathfrak{h}(\kappa) \leq \mathfrak{s}(\kappa) = \kappa^*$ and $\kappa^* = \mathfrak{t}(\kappa) \leq \mathfrak{h}(\kappa)$. Thus $\mathfrak{h}(\kappa) = \kappa^*$. $\square$

2.5. Open questions

1. Are there non-trivial upper bounds for $\mathfrak{i}(\kappa)$, when κ is an uncountable regular cardinal (possibly a large cardinal)?

2. Let κ be uncountable. Are there ZFC provable relations between $\mathfrak{i}(\kappa)$ and other generalized characteristics, that the classical independence number does not satisfy?
3. Is there a canonical forcing notion $\mathbb{P}$ (like the generalization of Hechler's poset adding a κ-mad family $\mathbb{H}_\kappa$) which adds a κ-maximal independent family?
4. A general question is whether we can separate not just a given cardinal invariant $\Phi(\kappa)$ from 2^κ, but more than one cardinal. For instance, it is consistent to have a model where $\kappa^+ < \mathfrak{b} < \mathfrak{a}$?
5. Let $\mathcal{I}$ be a κ-complete ideal on κ, say that we define a version of the distributivity number respect do $\mathcal{I}$ to be $\mathfrak{h}^{\mathcal{I}}(\kappa) = \min\{\lambda$: the quotient $\mathcal{P}(\kappa)/\mathcal{I}$ is not λ distributive $\}$. Is it possible to characterize when $\mathfrak{h}^{\mathcal{I}}(\kappa) > \kappa$? If $\mathcal{I}$ and $\mathcal{J}$ are two different κ-ideals, are the associated distributivity numbers different?

Part II.

A result on the countable case

Chapter 3

Three-dimensional iterations

In this chapter, we present the results obtained as joint work with Vera Fischer, Sy-David Friedman and Diego Mejía. Specifically, we provide a generalization of the method of matrix iterations, extending it to include a third dimension and as an application, we provide a generic extension where seven cardinals in the classical Cichoń's diagram are separated and the value of the almost disjointness number $\mathfrak{a}$ is decided. The main results of this chapter can be found in the paper *Coherent systems of finite support iterations* [Fis+16]. Unlike the two preceding chapters, this one focuses exclusively on cardinal invariants of the classical Baire space ω^ω.

The method of matrix iterations was introduced by Andreas Blass and Saharon Shelah in the paper *Ultrafilters with small generating sets* [BS89] to show the consistency of the inequality $\mathfrak{u} < \mathfrak{d}$. This method was later improved by Jörg Brendle and Vera Fischer in [BF11], where the terminology *matrix iteration* appeared for the first time. They showed that if κ and λ are arbitrary regular uncountable cardinals with $\kappa < \lambda$ then there is a cardinal preserving generic extension in which the inequality $\mathfrak{a} = \mathfrak{b} = \kappa < \mathfrak{s} = \lambda$ holds. Later, classical preservation properties for matrix iterations were improved by Diego Mejía [Mej13] to show several examples of models where the cardinals in Cichoń's diagram can be separated.

The motivation for extending the method of matrix iteration comes from trying to answer the following question: *Is it possible to decide the value of the cardinal invariant* $\mathfrak{a}$ *in classical models obtained as finite support iterations?* Most of the models we are interested in are models where we already have tools to decide the value of the unbounding number $\mathfrak{b}$ and since we have $\mathfrak{b} \leq \mathfrak{a}$, the question became whether we can decide the value of $\mathfrak{a}$ to be exactly $\mathfrak{b}$ in such models. Note that this requires to add a witness for $\mathfrak{a} = \mathfrak{b}$ and once we added it we have to ensure that it will be preserved upon the iteration of the forcings used to control the other invariants.

3.1. Matrix iterations

Since one of our goals is to generalize this method, in this section we include its basic theory and some applications. The idea is that the results presented here will motivate the general theory of coherent systems of finite support iterations that will be introduced in the subsequent sections.

3.1.1. Suslin posets and complete embeddability

We define the class of Suslin forcings and present some of its properties. The reason why is because this class has nice embeddability and absoluteness properties that we will use in the upcoming results. All the forcing notions we will use in this chapter belong to this class.

Definition 3.1 (Judah-Shelah in [IS88])**.** A poset $\mathbb{P} \subseteq \omega^\omega$ is *Suslin ccc* if it has the countable chain condition and in addition $\mathbb{P}$, the order relation $\leq$ and the incompatibility relation $\perp$ on $\mathbb{P}$ are analytic in the corresponding spaces they are subsets from (ω or $\omega \times \omega$).

One of the properties of Suslin ccc forcing notions that will be crucial is the following:

Remark 3.2. Let M, N be two models of ZFC and $\mathbb{P} \in M$ be a ccc Suslin poset, then the property "A is a maximal antichain" is absolute between M and N.

Proof. Let A be a maximal antichain in $\mathbb{P}$, since the forcing is ccc we can write $A = \{p_n : n \in \omega\}$, then the property A is maximal is equivalent to:

1. $\forall n \forall m (n \neq m \to p_n \perp p_m)$, which is clearly a $\mathbf{\Sigma}^1_1$ statement and,
2. $\forall q (q \in \mathbb{P} \vee \exists n (q \parallel p_n))$ which is $\mathbf{\Pi}^1_1$.

Thus $\mathbf{\Sigma}^1_1$ absoluteness gives us the result. □

Corollary 3.3. Let $M \subseteq N$ two models of ZFC and $\mathbb{P} \in M$ be a Suslin ccc poset, if G is $\mathbb{P}^N$-generic over N, then $G \cap M$ is $\mathbb{P}^M$-generic over M.

Definition 3.4 (Complete embeddability of forcings)**.** If $\mathbb{P}$ and $\mathbb{Q}$ are two forcing notions, a function $i : \mathbb{P} \to \mathbb{Q}$ is a *complete embedding* if:

- $\forall p_1, p_2 \in \mathbb{P}(p_1 \leq p_2 \to i(p_1) \leq i(p_2))$. We say that i is *order preserving*.
- $\forall p_1, p_2 \in \mathbb{P}(p_1 \perp p_2 \to i(p_1) \perp i(p_2))$. We say that *i preserves incompatibility*.
- For all $A \subseteq \mathbb{P}$: If A is a maximal antichain in $\mathbb{P}$, then $i(A)$ is also a maximal antichain in $\mathbb{Q}$.

When there is such i we write $\mathbb{P} \lessdot \mathbb{Q}$ and we say that $\mathbb{P}$ is a complete suborder of $\mathbb{Q}$ if in addition $\mathbb{P} \subseteq \mathbb{Q}$.

Definition 3.5. Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing notions that satisfy the first two items in the definition above, let also $q \in \mathbb{Q}$. We say that $p \in \mathbb{Q}$ is a *reduction of* q to $\mathbb{P}$ if and only if given $r \in \mathbb{P}$ such that $i(r) \perp q$ we have $r \perp p$.

The following lemma characterizes complete embeddability in terms of reductions.

Lemma 3.6. Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing notions such that there is an order preserving map $i : \mathbb{P} \to \mathbb{Q}$ which also preserves incompatibility. Then i is a complete embedding if and only if for every $q \in \mathbb{Q}$ there is a reduction of q to $\mathbb{P}$.

Proof. Assume first that we have reductions for conditions in $\mathbb{Q}$ and let A be a maximal antichain in $\mathbb{P}$. If $i(A)$ is not maximal in $\mathbb{Q}$ there exists a condition $q \in \mathbb{Q}$ incompatible with all the members of $i(A)$. Now take $p \in \mathbb{P}$ be a reduction of q to $\mathbb{P}$, since A is maximal, there exists $r \in A$ with $r \parallel p$, hence $i(r) \parallel q$ which is a contradiction.

For the other direction, fix $q \in \mathbb{Q}$ and take $A \subseteq \mathbb{P}$ to be a maximal set with respect to the following property: A is an antichain and $q \perp i(p)$ for all $p \in A$. Clearly $i(A)$ is an antichain in $\mathbb{Q}$, but so is $i(A) \cup \{q\}$. Using the hypothesis we have that A is not maximal in $\mathbb{P}$, so there is $r \in \mathbb{P}$ incompatible with all elements in A, then r is a reduction of q to $\mathbb{P}$. Otherwise there exists $s \in \mathbb{P}$ such that $i(s) \perp q$ and $s \parallel r$ which implies that the set $A \cup \{s\}$ has the same properties as A, and so the latest cannot be maximal. □

Definition 3.7 (Relative embeddability of forcings)**.** Let M be a transitive model of ZFC (or a finite large fragment of it). If $\mathbb{P}$ and $\mathbb{Q}$ are two forcing notions in M, a function $i : \mathbb{P} \to \mathbb{Q}$ is a *complete embedding respect to M* if:

- i is both order and incompatibility preserving.
- For all $A \subseteq \mathbb{P}$ with $A \in M$: If A is a maximal antichain in $\mathbb{P}$, then $i(A)$ is also a maximal antichain in $\mathbb{Q}$.

When there is such i we write $\mathbb{P} \lessdot_M \mathbb{Q}$.

Recall that in this case given $N \supseteq M$ another transitive model of ZFC with $\mathbb{Q} \in N$ and G is $\mathbb{Q}$-generic over N, then $G \cap \mathbb{P}$ is $\mathbb{P}$-generic over M and $M[G \cap \mathbb{P}] \subseteq N[G]$.

Moreover, if $\dot{\mathbb{P}}' \in M$ is a $\mathbb{P}$-name of a poset, $\dot{\mathbb{Q}}' \in N$ is a $\mathbb{Q}$-name of a poset and $\Vdash_{\mathbb{Q},N} \dot{\mathbb{P}}' \lessdot_{M^{\mathbb{P}}} \dot{\mathbb{Q}}'$, then $\mathbb{P} * \dot{\mathbb{P}}' \lessdot_M \mathbb{Q} * \dot{\mathbb{Q}}'$. In particular, if $M = N = V$, $\mathbb{P} * \dot{\mathbb{P}}' \lessdot \mathbb{Q} * \dot{\mathbb{Q}}'$ whenever $\mathbb{P} \lessdot \mathbb{Q}$ and $\Vdash_{\mathbb{Q}} \dot{\mathbb{P}}' \lessdot_{V^{\mathbb{P}}} \dot{\mathbb{Q}}'$.

The following results are due to Blass-Shelah (see [BS89]) and show how complete embeddability is an iterable property.

Lemma 3.8. Let $\mathbb{P}$ and $\mathbb{Q}$ partial orders and assume that $\mathbb{P} \lessdot \mathbb{Q}$. Let $\dot{\mathbb{A}}$ be a $\mathbb{P}$-name for a forcing notion and $\dot{\mathbb{B}}$ be a $\mathbb{Q}$-name for a forcing notion such that $\Vdash_{\mathbb{Q}} \dot{\mathbb{A}} \subseteq \dot{\mathbb{B}}$ and every maximal antichain of $\dot{\mathbb{A}}$ in $V^{\mathbb{P}}$ is a maximal antichain of $\dot{\mathbb{B}}$ in $V^{\mathbb{Q}}$. Then $\mathbb{P} * \dot{\mathbb{A}} \lessdot \mathbb{Q} * \dot{\mathbb{B}}$.

Proof. It is enough to show that every maximal antichain in $\mathbb{P} * \dot{\mathbb{A}}$ is maximal in $\mathbb{Q} * \dot{\mathbb{B}}$. Let $\{(p_\alpha, \dot{a}_\alpha) : \alpha < \kappa\} \subseteq \mathbb{P} * \dot{\mathbb{A}}$ be a maximal antichain and suppose towards a contradiction that there exists a condition $(q, \dot{b}) \in \mathbb{Q} * \dot{\mathbb{B}}$ such that for all $\alpha < \kappa$, $(q, \dot{b}) \perp (p_\alpha, \dot{a}_\alpha)$.

Take $\dot{H}$ to be the canonical $\mathbb{P}$-name for the $\mathbb{P}$-generic filter and let $\dot{A}$ to be a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}} \dot{A} = \{\alpha : p_\alpha \in \dot{H}\}$, we claim that $\Vdash_{\mathbb{P}} \{\dot{a}_\alpha : \alpha \in \dot{A}\}$ is a maximal antichain of $\dot{\mathbb{A}}$. Again, suppose towards a contradiction that there are a condition $p \in \mathbb{P}$ and a $\mathbb{P}$-name $\dot{a}$ for a condition in $\dot{\mathbb{A}}$ such that $p \Vdash \dot{a} \perp \dot{a}_\alpha$ for all $\alpha \in \dot{A}$. Then $(p, \dot{a}) \in \mathbb{P} * \dot{\mathbb{A}}$

and so there is $\alpha < \kappa$ such that $(p, \dot{a}) \parallel (p_\alpha, \dot{a}_\alpha)$ (because our assumption). Put $(r, \dot{c})$ be a common extension of them, then $r \Vdash \dot{c} \leq \dot{a}, \dot{a}_\alpha$ and since $r \leq p_\alpha$, r also forces $\alpha \in \dot{A}$, which is of course, a contradiction.

Finally, let G be a $\mathbb{Q}$-generic filter with $q \in G$. Because of the complete embeddability, there is a $\mathbb{P}$-generic filter H such that $V[H] \subseteq V[G]$. Put $b = \dot{b}[G]$, $a_\alpha = \dot{a}_\alpha[G] = \dot{a}_\alpha[H]$ and $A = \dot{A}[G] = \{\alpha < \kappa : p_\alpha \in H\}$, the argument above shows that $\{a_\alpha : \alpha \in A\}$ is a maximal antichain from $\mathbb{A}$ in $V[G]$, so there is $\alpha \in A$ with $b \parallel a_\alpha$.

Thus, there exists $s \in G$ such that $s \leq p_\alpha, q$ and $s \Vdash (\alpha \in \dot{A} \wedge \dot{b} \parallel \dot{a}_\alpha)$. This means that there is a $\mathbb{Q}$-name $\dot{d}$ such that $s \Vdash \dot{d} \leq \dot{b} \wedge \dot{d} \leq \dot{a}_\alpha$ which contradicts that our original set was maximal. □

Lemma 3.9. Let $(\mathbb{P}_{\alpha,n}, \dot{\mathbb{Q}}_{\alpha,n} : \alpha < \xi)$, $n \in \{0,1\}$ be finite support iterations such that for all $\alpha < \xi$, $\mathbb{P}_{\alpha,0} \lessdot \mathbb{P}_{\alpha,1}$. Then $\mathbb{P}_{\xi,0} \lessdot \mathbb{P}_{\xi,1}$.

Proof. Clearly $\mathbb{P}_{\xi,0} \subseteq \mathbb{P}_{\xi,1}$, let $q \in \mathbb{P}_{\xi,1}$ and take $\alpha < \xi$ to be such that $q \in \mathbb{P}_{\alpha,1}$, using the hypothesis $\mathbb{P}_{\alpha,0} \lessdot \mathbb{P}_{\alpha,1}$, we know that there is a reduction $p \in \mathbb{P}_{\alpha,0}$ of q to $\mathbb{P}_{\alpha,0}$. We claim that p is actually a reduction of q to $\mathbb{P}_{\xi,0}$: given $r \in \mathbb{P}_{\xi,0}$ such that $r \perp q$ we can write $r = r_0 \cup r_1$ where $r_0 \in \mathbb{P}_{\alpha,0}$ and $\mathrm{supp}(r_1) \in [\alpha, \xi)$, then $r_0 \perp q$ ($q \in \mathbb{P}_{\alpha,1}$) and since p is a reduction of q to $\mathbb{P}_{\alpha,0}$ we have $r_0 \perp p$ and this clearly implies $r \perp p$. □

The following properties relate Suslin forcing with the concepts introduced in the preliminaries chapter. Recall that given a Polish relational system $\mathcal{R} = (X, Y, \sqsubseteq)$, a poset $\mathbb{P}$ is *θ-$\sqsubset$-good* if for any $\mathbb{P}$-name $\dot{h}$ for a real in Y, there is a non-empty $H \subseteq Y$ of size $< \theta$ such that for any $x \in X$ that is $\mathcal{R}$-unbounded over H, $\Vdash x \not\sqsubseteq \dot{h}$ holds.

Lemma 3.10 ([Mej13, Theorem 7])**.** Let $\mathbb{P}$ be a Suslin ccc poset coded in M. If $M \models$ "$\mathbb{P}$ is $\mathcal{R}$-good", then in N, $\mathbb{P}^N$ forces that every real in $X \cap N$ which is $\mathcal{R}$-unbounded over M is $\mathcal{R}$-unbounded over $M^{\mathbb{P}^M}$.

Proof. Let $\dot{f} \in M$ be a $\mathbb{P}$-name for a real in ω^ω. Take $H \in M$ witnessing that $\mathbb{P}$ is $\mathcal{R}$-good, then:

$$\forall x \in \omega^\omega((\forall g \in H \neg(x \sqsubseteq g)) \to \Vdash \neg(x \sqsubseteq \dot{f}))(*)$$

Now, notice that this statement is absolute for M: the name $\dot{f}$ is fully decided by maximal antichains $(A_n : n \in \omega)$ where $A_n = \{p_{n,m} : m \in \omega\}$ and a sequence of partial functions $s^n_m \in \omega^{<\omega}$ such that $p_{n,m} \Vdash \dot{f} \upharpoonright n = s^n_m$. Let also $H = \{g_n : n \in \omega\}$ be an enumeration of H, then the statement $x \sqsubseteq \dot{f}$ is equivalent to:

$$\exists p \in \mathbb{P} \exists n \in \omega \forall k, m \in \omega (p \parallel p_{k,m} \to [s^k_m] \subseteq (\sqsubseteq_n)_x)$$

where $(\sqsubseteq_n)_x = \{g \in \omega^\omega : x \sqsubseteq_n g\}$. This implies that the statement $(*)$ is Π^1_1 and so absolute.

Then $N \models \forall x \in \omega^\omega(\forall g \in H(x \sqsubseteq \dot{f} \to x \sqsubseteq g))(*)$ and using that c is $\mathcal{R}$-unbounded in M, in particular we have that for all $g \in H$, $\neg(c \sqsubseteq g)$ and so, $\Vdash_{\mathbb{P},N} \neg(c \sqsubseteq \dot{f})$ as we wanted. □

Lemma 3.11 ([BF11, Lemmas 11 and 12]).

- Assume $\mathbb{P} \in M$ is a poset. Then, in N, $\mathbb{P}$ forces that every real in $X \cap N$ which is $\mathcal{R}$-unbounded over M is $\mathcal{R}$-unbounded over $M^{\mathbb{P}}$.
- Let $(\mathbb{P}_{l,n}, \dot{\mathbb{Q}}_{l,n} : l < \omega)$, $n \in \{0,1\}$ be finite support iterations such that for all $l < \omega$, $\mathbb{P}_{l,0} \lessdot \mathbb{P}_{l,1}$. Let also $V_{l,n} = V^{\mathbb{P}_{l,n}}$ and $c \in V_{0,1} \cap \omega^\omega$. If c is $\mathcal{R}$-unbounded over $V_{l,0}$ for all $l \in \omega$, then c is $\mathcal{R}$-unbounded over $V_{\omega,0}$ (seeing it as a real in $V_{\omega,1}$).

Proof. In both proofs we argue by contradiction:

- If $c \in N \cap X$ is not $\mathcal{R}$-unbounded over $M^{\mathbb{P}}$, then there exist a $\mathbb{P}$-name $\dot{f}$ for a real in ω^ω and a condition $p \in \mathbb{P}$ such that:

$$p \Vdash_{N,\mathbb{P}} (c \sqsubseteq \dot{f}).$$

  On the other hand, given $m \geq n$, there are conditions $q_m \leq p$ (in M) and partial functions $s_m \in \omega^{<\omega}$ such that $s_j \supseteq s_i$ when $j \geq i$ and $q_m \Vdash_{M,\mathbb{P}} \dot{f} \restriction m = s_m$. Then there exists $n \in \omega$ such that for all $m \geq n$, $q_m \Vdash [s_m] \subseteq (\sqsubseteq_m)_c$ where $(\sqsubseteq_m)_c = \{g \in \omega^\omega : c \sqsubseteq_m g\}$.

  Thus the function $f_0 \in \omega^\omega \cap M$ defined as $f_0 = \bigcup_{m \geq n} s_m$ belongs to M and $N \models c \sqsubseteq f_0$, which is a contradiction.

- Again, assume that $c \in V_{0,1} \cap X \subseteq V_{\omega,1} \cap X$ is not $\mathcal{R}$-unbounded over $V_{\omega,0}$, then as in the case above there are a $\mathbb{P}_{\omega,0}$-name (here we are using complete embeddability) $\dot{f}$ for a function in ω^ω, a condition $p \in \mathbb{P}_{\omega,1}$ and $n \in \omega$ such that:

$$p \Vdash \forall m \geq n (c \sqsubseteq_m \dot{f}).$$

  Since p has finite support $p \in \mathbb{P}_{l,1}$ for some $l < \omega$. Let $G_{l,1}$ to be a $\mathbb{P}_{l,1}$-generic filter containing p and $f' = \dot{f} \restriction G_{l,0}$ be the corresponding quotient name (here $G_{l,0} = G_{l,1} \cap \mathbb{P}_{l,0}$).

  Take $\mathbb{R}^i_{\omega,l}$ be the quotient poset $\mathbb{P}_{\omega,i}/G_{l,i}$, for $i \in \{0,1\}$ in $V[G_{l,i}] = V_{l,i}$. Then $f' \in V_{l,0}$ and for all $m \geq n$.

$$V_{l,1} \models \Vdash_{\mathbb{R}^1_{\omega,l}} (c \sqsubseteq_m f').$$

  In addition, for all $m \geq n$ we can find $q_m \in \mathbb{R}^0_{\omega,l}$ and $k_m \in \omega$ such that $q_m \Vdash f'(m) = k_m$. As in the case above define the following function:

$$f_0(j) = \begin{cases} 0 & \text{if } j < n \\ k_j & \text{if } j \geq n \end{cases}$$

  Then it is clear that $f_0 \in V_{l,0}$ and $V_{l,1} \models c \sqsubseteq f_0$, which is again a contradiction.

$\square$

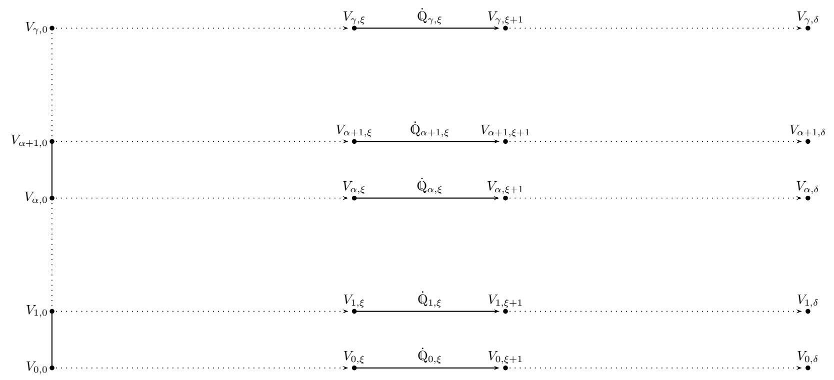

Figure 3.1.: Matrix of generic extensions.

Now, we give the general definition for a matrix iteration: simply, a matrix iteration is a linear system of relative completely embedded finite support iterations.

Definition 3.12 (Matrix iteration)**.** A *matrix iteration* $\mathbf{m}$ consists of:

1. Two ordinals $\delta^{\mathbf{m}}$ and $\gamma^{\mathbf{m}}$ (the dimensions of the matrix iteration, $\delta^{\mathbf{m}}$ represents the height of the iteration while $\gamma^{\mathbf{m}}$ represent its length).
2. An $\subseteq$-increasing sequence $(V^{\mathbf{m}}_{0,\beta} : \beta \leq \delta^{\mathbf{m}})$ of transitive models of ZFC [1] (the first column of models where the parallel iterations start) with $\delta^{\mathbf{m}}, \gamma^{\mathbf{m}} \in V^{\mathbf{m}}_{0,0}$.
3. Finite support iterations $(\mathbb{P}^{\mathbf{m}}_{\alpha,\beta}, \dot{\mathbb{Q}}^{\mathbf{m}}_{\alpha,\beta} : \alpha < \gamma^{\mathbf{m}})$ in $V^{\mathbf{m}}_{0,\beta}$ for $\beta \leq \delta^{\mathbf{m}}$ such that, for all $\beta \leq \beta' \leq \delta^{\mathbf{m}}$ and $\alpha < \gamma^{\mathbf{m}}$, $\mathbb{P}^{\mathbf{m}}_{\alpha,\beta} \lessdot_{V^{\mathbf{m}}_{0,\beta}} \mathbb{P}^{\mathbf{m}}_{\alpha,\beta'}$ and $\mathbb{P}^{\mathbf{m}}_{\alpha,\beta'}$ forces (in $V^{\mathbf{m}}_{0,\beta'}$) $\dot{\mathbb{Q}}^{\mathbf{m}}_{\alpha,\beta} \lessdot_{V^{\mathbf{m}}_{\alpha,\beta}} \dot{\mathbb{Q}}^{\mathbf{m}}_{\alpha,\beta'}$ where $V^{\mathbf{m}}_{\alpha,\beta} := (V^{\mathbf{m}}_{0,\beta})^{\mathbb{P}^{\mathbf{m}}_{\alpha,\beta}}$.

When $\mathbf{m}$ is clearly understood from the context, the upper index $\mathbf{m}$ will be omitted. From Lemma 3.11 it is clear that $\mathbb{P}_{\gamma,\beta} \lessdot_{V_{0,\beta}} \mathbb{P}_{\gamma,\beta'}$ for $\beta \leq \beta' \leq \delta$. Besides, $V_{\alpha,\beta} \subseteq V_{\alpha,\beta'}$ for any $\alpha \leq \gamma$. The idea of such a construction is to obtain a matrix $\langle V_{\alpha,\beta} : \alpha \leq \gamma, \beta \leq \delta \rangle$ of generic extensions as illustrated in Figure 3.1. We say that $\mathbf{m}$ is a *matrix iteration of ccc posets* when every $\dot{\mathbb{Q}}_{\alpha,\beta}$ is (forced to be) ccc.

3.1.2. An example

To illustrate the method of matrix iterations and how it is applied, we show an example in which six cardinals in Cichoń's diagram are separated. Through the argument, forcing notions as well as some preservation results presented in the preliminaries section are used. If the reader wants to have a more detailed presentation of these results, we refer it to [BJ95].

[1] or of a large enough finite fragment of it.

Theorem 3.13 (Theorem 21 in [Mej13]). *Let V be a model of* ZFC *and fix regular uncountable cardinals $\theta_0 \leq \theta_1 \leq \kappa \leq \mu$ and let $\lambda \geq \mu$ to be a cardinal with* $\mathrm{cof}(\lambda) > \theta_1$. *Then there is a cardinal preserving generic extension in which:* $\mathrm{add}(\mathcal{N}) = \theta_0$, $\mathrm{cov}(\mathcal{N}) = \theta_1$, $\mathfrak{b} = \mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \kappa$, $\mathfrak{d} = \mu$ *and* $\mathrm{non}(\mathcal{N}) = \mathfrak{c} = \lambda$.

Proof. Define a matrix iteration of ccc posets $\mathbf{m}$, $(\mathbb{P}^{\mathbf{m}}_{\alpha,\beta}, \dot{\mathbb{Q}}^{\mathbf{m}}_{\alpha,\beta} : \alpha < \gamma^{\mathbf{m}}, \beta \leq \delta^{\mathbf{m}})$ as follows: $\delta^{\mathbf{m}} = \mu$ and $\gamma^{\mathbf{m}} = \lambda \cdot \mu \cdot \kappa$. The first column of the matrix, that gives us the models $(V_{0,\beta} : \beta < \gamma^{\mathbf{m}})$ comes from a simple finite support iteration of length $\gamma^{\mathbf{m}}$ of Cohen forcing $\mathbb{C}$. Hence the model $V_{0,\beta}$ is a generic extension where we have added β-many Cohen reals. Additionally we will see the ordinal $\gamma^{\mathbf{m}}$ as the union of $\mu \cdot \kappa$-many intervals of order type λ, then we consider the following cases for every $\beta < \gamma^{\mathbf{m}}$.

- If $\beta = \lambda \cdot \xi$ for some $\xi < \mu \cdot \kappa$. Put $\dot{\mathbb{Q}}_{\alpha,\beta}$ to be a $\mathbb{P}_{\alpha,\beta}$-name for the forcing $\mathbb{E}$ (the eventually different forcing).
- If $\beta = \lambda \cdot \xi + 1$ for some $\xi < \mu \cdot \kappa$. Let $\dot{\mathbb{D}}_\beta$ to be a $\mathbb{P}_{t(\xi),\beta}$-name for the forcing $\mathbb{D}^{V_{t(\xi),\beta}}$ (Hechler forcing), where t is a function $t : \lambda \cdot \xi \to \mu$ such that t is onto and for every $\alpha < \mu$, $t^{-1}(\alpha) \cap \{\lambda \cdot \xi : \xi \in \mu \cdot \kappa\}$ is cofinal on $\mu \cdot \kappa$. Put:

$$\dot{\mathbb{Q}}^{\mathbf{m}}_{\alpha,\beta} = \begin{cases} \dot{\mathbb{D}}_\beta & \text{if } \alpha > t(\xi) \\ \mathbb{1} & \text{otherwise.} \end{cases}$$

For each $\xi < \mu \cdot \kappa$ and $\alpha < \mu$, let $(\dot{\mathbb{LOC}}^{\xi}_{\alpha,\gamma} : \gamma < \lambda)$ be an enumeration of all $\mathbb{P}_{\alpha,\lambda\cdot\xi+1}$-names for all σ-linked subposets of $\mathbb{LOC}^{V_{\alpha,\lambda\cdot\xi+1}}$ of size $< \theta_0$. Analogously let $(\dot{\mathbb{B}}^{\xi}_{\alpha,\gamma} : \gamma < \lambda)$ be an enumeration of all $\mathbb{P}_{\alpha,\lambda\cdot\xi+1}$-names for all subalgebras of $\mathbb{B}^{V_{\alpha,\lambda\cdot\xi+1}}$ of size $< \theta_1$. Let also $g : \lambda \to \mu \times \lambda$ be a bijection. Consider the following cases:

- If $\beta = \lambda \cdot \xi + 2 + 2 \cdot \rho$ for some $\xi < \mu \cdot \kappa$ and $\rho < \lambda$ and put $\dot{\mathbb{Q}}^{\mathbf{m}}_{\alpha,\beta} = \dot{\mathbb{LOC}}^{\xi}_{g(\rho)}$.
- Finally, if $\beta = \lambda \cdot \xi + 2 + 2 \cdot \rho + 1$ for some $\xi < \mu \cdot \kappa$ and $\rho < \lambda$ put $\dot{\mathbb{Q}}^{\mathbf{m}}_{\alpha\beta} = \dot{\mathbb{B}}^{\xi}_{g(\rho)}$

Now, look at the $V^{\mathbb{P}} = V^{\mathbb{P}_{\delta^{\mathbf{m}},\gamma^{\mathbf{m}}}}$-generic extension. Our goal now is to show, that in this generic extension the equalities in our theorem hold. First, we need to prove the following lemma:

Claim 3.14.

- Let $\beta \leq \gamma^{\mathbf{m}}$, given $p \in \mathbb{P}_{\delta^{\mathbf{m}},\beta}$ there exists $\alpha < \delta^{\mathbf{m}}$ such that $p \in \mathbb{P}_{\alpha,\beta}$.
- Let $\dot{f}$ be a $\mathbb{P}_{\delta^{\mathbf{m}},\beta}$-name for a real, then there is $\alpha < \delta^{\mathbf{m}}$ such that $\dot{f}$ is a $\mathbb{P}_{\alpha,\beta}$-name.

Proof of the claim: This is a similar argument to the one in [BS89], we will just adapt it to this specific iteration: we prove both statements simultaneously by induction on $\beta \leq \gamma^{\mathbf{m}}$ noticing that the second one follows from the first one (our posets are all ccc). If $\beta = 0$ we have a simple Cohen iteration, which satisfies the property. If β is a limit ordinal and $p \in \mathbb{P}_{\delta^{\mathbf{m}},\beta}$, since p has finite support, there is $\beta' < \beta$ such that $p \in \mathbb{P}_{\delta^{\mathbf{m}},\beta}$ and using the induction hypothesis we find a coordinate $\alpha < \delta^{\mathbf{m}}$ with $p \in \mathbb{P}_{\alpha,\beta'}$.

Finally suppose that $\beta = \beta' + 1$ is a successor ordinal and $p \in \mathbb{P}_{\delta^{\mathbf{m}},\beta}$, then $p = (q, \dot{r})$, where $q \in \mathbb{P}_{\delta^{\mathbf{m}},\beta'}$ and $q \Vdash \dot{r} \in \mathbb{Q}_{\delta^{\mathbf{m}},\beta'}$. Using the induction hypotheses there exists $\alpha' < \delta^{\mathbf{m}}$ with $q \in \mathbb{P}_{\alpha',\beta'}$, in addition $\dot{r}$ can be actually seeing as a $\mathbb{P}_{\alpha'',\beta}$-name. Note that depending on the step, either $\dot{r}$ is a name for a condition in the eventually different forcing or in Hechler forcing or in a σ-linked subposets of the localization forcing or in a subalgebra of random forcing or is trivial. In any case, the condition $\dot{r}$ depends on some small set of $\mathbb{P}_{\delta^{\mathbf{m}},\beta'}$-names for reals. For instance, a condition in $\mathbb{D}$ has the form $(s, \dot{f})$, where $\dot{f}$ is a name for a function in ω^ω. Hence, we can use the induction hypothesis from the second statement to find the α'' mentioned above. Thus, we conclude $p \in \mathbb{P}_{\eta,\beta}$ where $\eta = \sup\{\alpha', \alpha''\}$. $\square$

Continuation of the proof of Theorem 3.13: We give arguments to explain why the cardinal invariants in this model are as we wanted.

- $\mathrm{add}(\mathcal{N}) = \theta_0$: For the inequality $\mathrm{add}(\mathcal{N}) \leq \theta_0$, take $X \subseteq \omega^\omega \cap V^{\mathbb{P}}$ of size $< \theta_0$, then using the claim we have that there is a pair $(\alpha, \beta = \lambda \cdot \xi + 1) \in (\delta^{\mathbf{m}}, \gamma^{\mathbf{m}})$ such that $X \in V_{\alpha,\beta}$. Then, in the model $V_{\alpha,\beta}$ there is a transitive model N of (a sufficiently large fragment of) ZFC such that $X \subseteq N$ and $|N| < \theta_0$, hence there is an ordinal $\gamma < \lambda$ with the property $\mathrm{LOC}^{\xi}_{\alpha,\gamma} = \mathbb{LOC}^N$, let then $\rho = g^{-1}(\alpha, \gamma)$ and $\beta' = \lambda \cdot \xi + 2 + 2\rho$. The forcing $\dot{\mathbb{Q}}^{\mathbf{m}}_{\alpha,\beta'}$ adds a slalom that captures all the reals in X, so the inequality follows. On th other hand $\mathrm{add}(\mathcal{N}) \leq \theta_0$ follows because since all forcing notions in our iteration are σ-centered we have in particular that they are $\theta_0 - \mathrm{LOC}$-good and Theorem 0.26 gives us $\mathrm{add}(\mathcal{N}) = \mathfrak{b}(\mathrm{LOC}) \leq \theta_0$.
- $\mathrm{cov}(\mathcal{N}) = \theta_1$. The proof is bit similar to the one above, to prove the inequality $\mathrm{cov}(\mathcal{N}) \leq \theta_1$, take a family $\mathcal{X}$ of Borel null sets coded in $V^{\mathbb{P}}$ of size $< \theta_1$, then again using the claim we have that there is a pair $(\alpha, \beta = \lambda \cdot \xi + 1) \in (\delta^{\mathbf{m}}, \gamma^{\mathbf{m}})$ such that all the sets in $\mathcal{X}$ are coded in $V_{\alpha,\beta}$. Then there is an ordinal $\gamma < \lambda$ such that, the generic real added by $\mathbb{B}^{\xi}_{\alpha,\gamma}$ avoids all the null sets in $\mathcal{X}$, let then $\rho = g^{-1}(\alpha, \gamma)$ and $\beta' = \lambda \cdot \xi + 2 + 2\rho + 1$. The forcing $\dot{\mathbb{Q}}^{\mathbf{m}}_{\alpha,\beta'} = \mathbb{B}^{\xi}_{\alpha,\gamma}$ and the inequality holds. On the other hand $\mathrm{cov}(\mathcal{N}) \leq \theta_1$ follows because since all forcing notions in our iteration are σ-centered we have in particular, they are $\theta_1 - \mathcal{E}\lceil_b$-good and Theorem 0.26 gives us $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{b}(\mathcal{E}\lceil_b) \leq \theta_1$.
- $\mathfrak{b} = \mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \kappa$. We prove first $\mathfrak{b} \geq \kappa$: Take $\mathcal{F}$ be a family of reals in $V^{\mathbb{P}}$ of size $< \kappa$ and find again $(\alpha, \beta = \lambda \cdot \xi)$ such that $\mathcal{F} \in V_{\alpha,\beta}$. We use the restricted dominating reals $\{d_\xi : \xi < \mu \cdot \kappa\}$ added at steps were we used Hechler forcing, since $t^{-1}(\alpha)$ is cofinal on $\lambda \cdot \xi$, there exists $\rho \in [\xi, \mu \cdot \kappa)$ such that the real $\dot{d}_\rho$ added by $\mathbb{Q}_{\alpha+1,\rho'}$ where $\rho' = \lambda \cdot \rho + 1$ dominates all the reals in $\mathcal{F}$.

  Now, the inequality $\mathrm{cov}(\mathcal{M}) \leq \kappa$ follows from the cofinally κ-many eventually different reals added at the first iteration in each of the μ-many intervals and $\mathrm{non}(\mathcal{M}) \leq \kappa \leq \mathrm{cov}(\mathcal{M})$ follows from the family of κ-many Cohen reals added by any finite support iteration.
- $\mathfrak{d} = \mu$. The family of dominating reals mentioned on the item above is a witness for $\mathfrak{d} \leq |\mu \cdot \kappa| = \mu$, on the other hand $\mathfrak{d} \geq |\mu \cdot \kappa| = \mu$ use the Cohen functions added

in the first part of the iteration, this family has size μ and will remain unbounded in the final generic extension (see Theorem 3.19).

- $\mathrm{non}(\mathcal{N}) = \mathfrak{c} = \lambda$. Finally recall Theorem 0.26 to see that $\mathrm{non}(\mathcal{N}) \geq \mathfrak{d}(\mathcal{E}\restriction_b) \geq |\gamma^{\mathbf{m}}| = \lambda$ and $\mathfrak{c} \leq \lambda$ follow because $|\mathbb{P}| = \lambda$.

$\square$

Note: In the construction of the matrix of generic extensions described above we lay emphasis on the fact that, at some steps of the iteration we are taking what we will call *full-generics*, this refers specifically to the steps where we consider the eventually different forcing $\mathbb{E}$. Note that if we are in a step of the form (α, β) where $\beta = \lambda \cdot \xi$ we force with $\mathbb{E}$ of the corresponding model $V^{\mathbb{P}_{\alpha,\beta}}$, and this works for every $\alpha < \mu$. On the other hand, at steps of the form $\lambda \cdot \xi + 1$ either we force with the trivial forcing or we use the same $\mathbb{P}_{t(\xi),\beta}$-name for Hechler forcing respect to some specific model (chosen with the help of the function t) and force with the same forcing along all $\alpha > t(\xi)$, in this specific case we will say that we forced a *partial generic*.

This idea motivates the definitions we will give in the next sections which at first sight might seem abstract. However, if the reader has in mind the example above and what is the difference between taking full or partial generics, the definitions will be clearer.

3.2. Coherent systems of finite support iterations

The definition of a coherent system of finite support iterations we introduce now aims to provide a general framework that will include matrix iterations as a specific case and in addition, gives us the freedom to add more dimensions to iterate.

Definition 3.15. A *coherent system of finite support iterations* $\mathbf{s}$ is composed by the following objects:

1. A partially ordered set $I^{\mathbf{s}}$ and an ordinal $\pi^{\mathbf{s}}$.
2. A system of posets $(\mathbb{P}^{\mathbf{s}}_{i,\xi} : i \in I^{\mathbf{s}}, \xi \leq \pi^{\mathbf{s}})$ such that:
   a) $\mathbb{P}^{\mathbf{s}}_{i,0} \lessdot \mathbb{P}^{\mathbf{s}}_{j,0}$ whenever $i \leq j$ in $I^{\mathbf{s}}$, and
   b) $\mathbb{P}^{\mathbf{s}}_{i,\eta}$ is the direct limit of $(\mathbb{P}^{\mathbf{s}}_{i,\xi} : \xi < \eta)$ for each limit $\eta \leq \pi^{\mathbf{s}}$.
3. A sequence $(\dot{\mathbb{Q}}^{\mathbf{s}}_{i,\xi} : i \in I^{\mathbf{s}}, \xi < \pi^{\mathbf{s}})$ where each $\dot{\mathbb{Q}}^{\mathbf{s}}_{i,\xi}$ is a $\mathbb{P}^{\mathbf{s}}_{i,\xi}$-name for a poset, $\mathbb{P}^{\mathbf{s}}_{i,\xi+1} = \mathbb{P}^{\mathbf{s}}_{i,\xi} * \dot{\mathbb{Q}}^{\mathbf{s}}_{i,\xi}$ and $\mathbb{P}^{\mathbf{s}}_{j,\xi}$ forces $\dot{\mathbb{Q}}^{\mathbf{s}}_{i,\xi} \lessdot_{V^{\mathbb{P}^{\mathbf{s}}_{i,\xi}}} \dot{\mathbb{Q}}^{\mathbf{s}}_{j,\xi}$ whenever $i \leq j$ belong to $I^{\mathbf{s}}$ and $\mathbb{P}^{\mathbf{s}}_{i,\xi} \lessdot \mathbb{P}^{\mathbf{s}}_{j,\xi}$.

Note that, for a fixed $i \in I^{\mathbf{s}}$, the posets $\langle \mathbb{P}^{\mathbf{s}}_{i,\xi} : \xi \leq \pi^{\mathbf{s}} \rangle$ are generated by finite support iterations $(\mathbb{P}'_{i,\xi}, \dot{\mathbb{Q}}'_{i,\xi} : \xi < 1 + \pi^{\mathbf{s}})$ where $\dot{\mathbb{Q}}'_{i,0} = \mathbb{P}^{s}_{i,0}$ and $\dot{\mathbb{Q}}'_{i,1+\xi} = \dot{\mathbb{Q}}^{\mathbf{s}}_{i,\xi}$ for all $\xi < 1 + \pi^{\mathbf{s}}$. Therefore (by induction) $\mathbb{P}'_{i,1+\xi} = \mathbb{P}_{i,\xi}$ for all $\xi \leq \pi^{s}$ and, thus, $\mathbb{P}^{\mathbf{s}}_{i,\xi} \lessdot \mathbb{P}^{\mathbf{s}}_{i,\eta}$ whenever $\xi \leq \eta \leq \pi^{\mathbf{s}}$.

On the other hand, by Lemmas 3.8 and 3.9, $\mathbb{P}^{\mathbf{s}}_{i,\xi} \lessdot \mathbb{P}^{\mathbf{s}}_{j,\xi}$ whenever $i \leq j$ in $I^{\mathbf{s}}$ and $\xi \leq \pi^{\mathbf{s}}$.

For $j \in I^{\mathbf{s}}$ and $\eta \leq \pi^{\mathbf{s}}$ we write $V^{\mathbf{s}}_{j,\eta}$ for the $\mathbb{P}^{\mathbf{s}}_{j,\eta}$-generic extension. Concretely, if G is $\mathbb{P}^{\mathbf{s}}_{j,\eta}$-generic over V, $V^{\mathbf{s}}_{j,\eta} := V[G]$ and $V^{\mathbf{s}}_{i,\xi} := V[\mathbb{P}^{\mathbf{s}}_{i,\xi} \cap G]$ for all $i \leq j$ in $I^{\mathbf{s}}$ and $\xi \leq \eta$. Clearly $V^{\mathbf{s}}_{i,\xi} \subseteq V^{\mathbf{s}}_{j,\eta}$.

We say that the coherent system $\mathbf{s}$ has the *ccc* if, additionally, $\mathbb{P}^{\mathbf{s}}_{i,0}$ has the ccc and $\mathbb{P}^{\mathbf{s}}_{i,\xi}$ forces that $\dot{\mathbb{Q}}^{\mathbf{s}}_{i,\xi}$ has the ccc for each $i \in I^{\mathbf{s}}$ and $\xi < \pi^{\mathbf{s}}$. This implies that $\mathbb{P}^{\mathbf{s}}_{i,\xi}$ has the ccc for all $i \in I^{\mathbf{s}}$ and $\xi \leq \pi^{\mathbf{s}}$.

For our applications, we consider the following particular cases of coherent systems:

- $I^{\mathbf{s}}$ is a well-ordered set, we say that $\mathbf{s}$ is a *2D-coherent system of finite support iterations)*. Note, that this is exactly the case of a matrix iteration.
- If $I^{\mathbf{s}}$ is of the form $\{i_0, i_1\}$ ordered as $i_0 < i_1$, we say that $\mathbf{s}$ is a *coherent pair of finite support iterations*.
- If $I^{\mathbf{s}} = \gamma^{\mathbf{s}} \times \delta^{\mathbf{s}}$ where $\gamma^{\mathbf{s}}$ and $\delta^{\mathbf{s}}$ are ordinals and the order of $I^{\mathbf{s}}$ is defined as $(\alpha, \beta) \leq (\alpha', \beta')$ if and only if $\alpha \leq \alpha'$ and $\beta \leq \beta'$, we say that $\mathbf{s}$ is a *3D-coherent system of finite support iterations.*

Notation:

Given a coherent system $\mathbf{s}$ and a set $J \subseteq I^{\mathbf{s}}$, $\mathbf{s}|J$ denotes the coherent system with $I^{\mathbf{s}|J} = J$, $\pi^{\mathbf{s}|J} = \pi^{\mathbf{s}}$ and the posets and names corresponding to (2.) and (3.) in Definition 3.15 are the corresponding ones coming from $\mathbf{s}$.

On the other hand, if $\eta \leq \pi^{\mathbf{s}}$, $\mathbf{s}\restriction\eta$ denotes the coherent system with $I^{\mathbf{s}\restriction\eta} = I^{\mathbf{s}}$, $\pi^{\mathbf{s}\restriction\eta} = \eta$ and the posets for (2.) and (3.) defined likewise from $\mathbf{s}$. Note that, if $i_0 < i_1$ in $I^{\mathbf{s}}$, then $\mathbf{s}|\{i_0, i_1\}$ is a coherent pair and $\mathbf{s}|\{i_0\}$ corresponds to the finite support iteration $(\mathbb{P}'_{i_0,\xi}, \dot{\mathbb{Q}}'_{i_0,\xi} : \xi < 1 + \pi^{\mathbf{s}})$.

If $\mathbf{t}$ is a 3D-coherent system, for $\alpha < \gamma^{\mathbf{t}}$, $\mathbf{t}_\alpha := \mathbf{t}|\{(\alpha, \beta) : \beta < \delta^{\mathbf{t}}\}$ which is a 2D-coherent system where $I^{\mathbf{t}_\alpha}$ has order type $\delta^{\mathbf{t}}$. For $\beta < \delta^{\mathbf{t}}$, $\mathbf{t}^\beta := \mathbf{t}|\{(\alpha, \beta) : \alpha < \delta^{\mathbf{t}}\}$ which is a 2D-coherent system where $I^{\mathbf{t}^\delta}$ has order type $\gamma^{\mathbf{t}}$. The indexes $\mathbf{s}$ are omitted when there is no place for ambiguity.

Look back to the proof of Theorem 3.13. There were two important facts that were used in the proof and which we want to generalize to our new construction of coherent systems of finite support iterations, particularly to the 3D-case. Recall the following:

1. For $\alpha < \gamma$, there is a real $c_\alpha \in V_{\alpha+1,0}$ which is unbounded over V_{α_0} and remains unbounded over all the models in the α-th row along the coherent pair $\mathbf{m} \restriction \{\alpha, \alpha + 1\}$ (see Lemma 3.11).
2. Assume that $\gamma^{\mathbf{m}}$ has uncountable cofinality. Given any column of the matrix, any real in the model of the top is actually in some of the models below (see Claim 3.14).

In addition, Claim 3.14 can be reformulated in the context on coherent systems as follows:

Lemma 3.16. Let $\mathbf{m}$ be a ccc 2D-coherent system with $I^{\mathbf{m}} = \gamma + 1$ an ordinal and $\pi^{\mathbf{m}} = \pi$. Assume that

(i) γ has uncountable cofinality,

(ii) $\mathbb{P}_{\gamma,0}$ is the direct limit of $(\mathbb{P}_{\alpha,0} : \alpha < \gamma)$, and

(iii) for any $\xi < \pi$, $\mathbb{P}_{\gamma,\xi}$ forces "$\dot{\mathbb{Q}}_{\gamma,\xi} = \bigcup_{\alpha<\gamma} \dot{\mathbb{Q}}_{\alpha,\xi}$" whenever $\mathbb{P}_{\gamma,\xi}$ is the direct limit of $(\mathbb{P}_{\alpha,\xi} : \alpha < \gamma)$.

Then, for any $\xi \leq \pi$, $\mathbb{P}_{\gamma,\xi}$ is the direct limit of $(\mathbb{P}_{\alpha,\xi} : \alpha < \gamma)$.

Corollary 3.17 ([BF11, Lemma 15]). If $\mathbf{m}$ is a standard 2D-coherent system with $I^{\mathbf{m}} = \gamma + 1$ and ordinal and $\pi^{\mathbf{m}} = \pi$ satisfying (1) and (2) of the Lemma above then, for any $\xi \leq \pi$, $\mathbb{P}_{\gamma,\xi}$ is the direct limit of $(\mathbb{P}_{\alpha,\xi} : \alpha < \gamma)$.

Remember that a matrix iteration is basically a system of linear iterations. Analogously, a 3D-coherent system is simply a system of 2D-coherent systems, and so on. In the section about 3D- systems we will show that the properties we want to have are "inherited " by the properties that a 2D-coherent system already has.

For our applications to constellations of Cichoń's diagram, the following type of coherent systems will be crucial. To motivate the definition we ask the reader to go back to the discussion on *full generics* and *restricted generics*.

Definition 3.18. A coherent system of finite support iterations $\mathbf{s}$ is *standard* if

1. it consists, additionally, of:
   a) A partition $\langle S^{\mathbf{s}}, C^{\mathbf{s}} \rangle$ of $\pi^{\mathbf{s}}$.
   b) A function $\Delta^{\mathbf{s}} : C^{\mathbf{s}} \to I^{\mathbf{s}}$ so that $\Delta^{\mathbf{s}}(i)$ is not maximal in $I^{\mathbf{s}}$ for all $i \in C^{\mathbf{s}}$.
   c) A sequence $\langle \mathbb{S}^{\mathbf{s}}_{\xi} : \xi \in S^{\mathbf{s}} \rangle$ where each $\mathbb{S}^{\mathbf{s}}_{\xi}$ is either a Suslin ccc poset or a random algebra.
   d) A sequence $\langle \dot{\mathbb{Q}}^{\mathbf{s}}_{\xi} : \xi \in C^{\mathbf{s}} \rangle$ such that each $\dot{\mathbb{Q}}^{\mathbf{s}}_{\xi}$ is a $\mathbb{P}^{\mathbf{s}}_{\Delta^{\mathbf{s}}(\xi),\xi}$-name of a poset which is forced to be ccc by $\mathbb{P}^{\mathbf{s}}_{i,\xi}$ for all $i \geq \Delta^{\mathbf{s}}(\xi)$ in $I^{\mathbf{s}}$, and
2. it is satisfied, for any $i \in I^{\mathbf{s}}$ and $\xi < \pi^{\mathbf{s}}$, that

$$\dot{\mathbb{Q}}^{\mathbf{s}}_{i,\xi} = \begin{cases} (\mathbb{S}^{\mathbf{s}}_{\xi})^{V^{\mathbf{s}}_{i,\xi}} & \text{if } \xi \in S^{\mathbf{s}} \\ \dot{\mathbb{Q}}^{\mathbf{s}}_{\xi} & \text{if } \xi \in C^{\mathbf{s}} \text{ and } i \geq \Delta^{\mathbf{s}}(\xi), \\ \mathbb{1} & \text{otherwise.} \end{cases}$$

A standard coherent system as above is constructed by using posets adding generic reals and the cases whether $\xi \in S$ or $\xi \in C$ indicate the steps in which we choose either a *full* or a *restricted* generic. Namely, in the first case if $\xi \in S$, $\mathbb{S}_{\xi}$ adds a real that is generic over $V_{i,\xi}$ for all $i \in I$; on the second case, $\xi \in C$ and $\dot{\mathbb{Q}}_{\xi}$ adds a real adds a real which is generic over $V_{\Delta(\xi),\xi}$ but not necessarily over $V_{i,\xi}$ when $i > \Delta(\xi)$.

The elements of the above sections dealing with matrix iteration can be summarized in the following result.

Theorem 3.19 ([Mej13, Theorem 10 & Corollary 1]). *Let* $\mathbf{m}$ *be a standard 2D-coherent system with* $I^{\mathbf{m}} = \gamma + 1$ *(an ordinal),* $\pi^{\mathbf{m}} = \pi$ *and* $\mathcal{R} = \langle X, Y, \sqsubset \rangle$ *a Polish relational system coded in* V. *Assume that*

1. *for any* $\xi \in S$ *and* $\alpha \leq \gamma$, $\mathbb{P}_{\alpha,\xi}$ *forces that* $\dot{\mathbb{Q}}_{\alpha,\xi} = \mathbb{S}_{\xi}^{V_{\alpha,\xi}}$ *is* $\mathcal{R}$*-good and*
2. *for any* $\alpha < \gamma$ *there is a* $\mathbb{P}_{\alpha+1,0}$*-name* $\dot{c}_\alpha$ *of a* $\mathcal{R}$*-unbounded member of* X *over* $V_{\alpha,0}$.

Then, for any $\xi \leq \pi$ *and* $\alpha < \gamma$, $\mathbb{P}_{\alpha+1,\xi}$ *forces that* $\dot{c}_\alpha$ *is* $\mathcal{R}$*-unbounded over* $V_{\alpha,\xi}$. *In addition, if* $\mathbf{m}$ *satisfies (i) and (ii) of Lemma 3.16 then* $\mathbb{P}_{\gamma,\pi}$ *forces* $\mathfrak{b}(\mathcal{R}) \leq \mathrm{cf}(\gamma) \leq \mathfrak{d}(\mathcal{R})$.

Proof. The first statement is a direct consequence of Lemmas 3.10, and 3.11. For the second statement, note that Corollary 3.17 implies that, in $V_{\gamma,\pi}$, $\{c_{\alpha_\eta} : \eta < \mathrm{cf}(\gamma)\}$ is a $\mathrm{cf}(\gamma)$-$\mathcal{R}$-unbounded family where $\langle \alpha_\eta : \eta < \mathrm{cf}(\gamma) \rangle \in V$ is an increasing cofinal sequence of γ, so $\mathfrak{b}(\mathcal{R}) \leq \mathrm{cf}(\gamma) \leq \mathfrak{d}(\mathcal{R})$ follows. $\square$

3.3. Preservation of Hechler mad families

Keep in mind, that one of our goals is to decide the value of the almost disjointness number $\mathfrak{a}$ to be equal to $\mathfrak{b}$ in models where also cardinal invariants in Cichoń's diagram are computed. To achieve this goal, this section presents firstly, how to add a specific type of maximal almost disjoint family along a finite support iteration. Secondly, we review the existing results on preservation theory for such families and finally, we present extensions of this results to a broader class of forcings.

Definition 3.20 (Hechler mad families [Hec72]). For a set Ω define the poset $\mathbb{H}_\Omega := \{p : F_p \times n_p \to 2 : F_p \in [\Omega]^{<\aleph_0}$ and $n_p < \omega\}$. The order is given by $q \leq p$ if and only if $p \subseteq q$ and, for any $i \in n_q \setminus n_p$, there is at most one $z \in F_p$ such that $q(z,i) = 1$.

If G is $\mathbb{H}_\Omega$-generic over V then $\mathcal{A} = A_G := \{A_z : z \in \Omega\}$ is an almost disjoint family where $a_z \subseteq \omega$ is defined as $i \in A_z$ if and only if $p(z,i) = 1$ for some $p \in G$. Moreover, $V[G] = V[A]$ and when Ω is uncountable $\mathcal{A}$ is also maximal in $V[G]$.

The following results correspond to the study that Jörg Brendle and Vera Fischer [BF11] did from this kind of mad families and how to guarantee that they are preserved after forcing with some specific posets.

Note that, if $\Omega \subseteq \Omega'$ it is clear that $\mathbb{H}_\Omega \lessdot \mathbb{H}_{\Omega'}$ and the quotient $\mathbb{H}_{\Omega'}/\mathbb{H}_\Omega$ is nicely expressed: Let G be a $\mathbb{H}_\Omega$-generic filter, then in $V[G]$ let $\mathbb{H}_{\Omega'\setminus\Omega}$ be the poset of all pairs (p, H) such that $p : F_p \times n_p$ with $F_p \in [\Omega' \setminus \Omega]^{<\omega}$, $n_p \in \omega$ and $H \subseteq [\Omega]^{<\omega}$ with the order defined by $(q, K) \leq (p, H)$ if and only if $q \leq_{\mathbb{H}_{\Omega'}}$, $H \subseteq K$ and for $w \in F_p, v \in H$ and $i \in n_q \setminus n_p$ if $i \in A_v$ then $q(w, i) = 0$. Then it holds that $\mathbb{H}_{\Omega'} = \mathbb{H}_\Omega * \dot{\mathbb{H}}_{\Omega'\setminus\Omega}$.

Furthermore, if $\mathcal{C}$ is a $\subseteq$-chain of sets then $\mathbb{H}_{\bigcup \mathcal{C}} = \mathrm{limdir}_{\Omega \in \mathcal{C}} \mathbb{H}_\Omega$. Therefore, if γ is an ordinal, $\mathbb{H}_\gamma$ can be obtained by a finite support iteration of length γ where $\mathbb{H}_\alpha$ is the poset obtained in the α-th stage of the iteration and the quotient $\mathbb{H}_{\alpha+1}/\mathbb{H}_\alpha$, which is σ-centered, is the α-th iterand. Since $\mathbb{H}_\Omega$ only depends on the size of Ω then $\mathbb{H}_\Omega$ has precaliber ω_1. Moreover, if Ω is non-empty and countable then $\mathbb{H}_\Omega \simeq \mathbb{C}$ and if $|\Omega| = \aleph_1$, then $\mathbb{H}_\Omega \simeq \mathbb{C}_{\omega_1}$.

From now on, fix transitive models of ZFC, M and N with $M \subseteq N$.

Definition 3.21 ([BF11, Definition 2]). Let $\mathcal{A} = (A_z : z \in \Omega) \in M$ be a family of infinite subsets of ω and $a^* \in [\omega]^\omega$ (not necessarily in M). Say that a^* *diagonalizes M outside $\mathcal{A}$* if for all $h \in M$, $h : \omega \times [\Omega]^{<\omega} \to \omega$ and for any $m < \omega$, there are $i \geq m$ and $F \in [\Omega]^{<\omega}$ such that $[i, h(i, F)) \setminus \bigcup_{z \in F} A_z \subseteq a^*$.

Given a collection A of subsets of ω, *the ideal generated by $\mathcal{A}$* is defined as

$$\mathcal{I}(\mathcal{A}) := \{x \subseteq \omega : x \subseteq^* \bigcup_{A \in F} A \text{ for some finite } F \subseteq \mathcal{A}\}.$$

Lemma 3.22 ([BF11, Lemma 3]). If a^* diagonalizes M outside $\mathcal{A}$ then $|a^* \cap x| = \aleph_0$ for any $x \in M \setminus \mathcal{I}(\mathcal{A})$.

Proof. If not, there exists $m \in \omega$ such that $a^* \cap X \subseteq m$. Let $i \geq m$ and $F \in [\Omega]^{<\omega}$, since $x \in M \setminus \mathcal{I}(\mathcal{A})$, there is $F \subseteq \mathcal{A}$ finite such that $x \not\subseteq^* \bigcup_{A \in F} A$ and we can find $k_{i,F} \geq i$ with $k_{i,F} \in x \setminus \bigcup_{A \in F} A$. Put h to be the function with domain $\omega \times [\Omega]^{<\omega}$ defined by:

$$h(i, F) = \begin{cases} k_{i,F} + 1 & \text{if } i \geq m \\ 0 & \text{otherwise} \end{cases}$$

Clearly, h so defined is a function in M such that $[i, h(i, F)) \setminus \bigcup_{A \in F} A \not\subseteq a^*$ for all $i \geq m$ which contradicts the diagonalization property of a^*. $\square$

Lemma 3.23 ([BF11, Lemma 4]). Let Ω be a set, $z^* \in \Omega$ and $\mathcal{A} := \{A_z : z \in \Omega\}$ the almost disjoint family added by $\mathbb{H}_\Omega$. Then $\mathbb{H}_\Omega$ forces that A_{z^*} diagonalizes $V^{\mathbb{H}_{\Omega \setminus \{z^*\}}}$ outside $\mathcal{A}{\restriction}(\Omega \setminus \{z^*\})$.

Proof. Let G be an $\mathbb{H}_\Omega$-generic filter and $h : \omega \times [\Omega]^{<\omega} \to \omega \in V[G_{z^*}]$ (here $G_{z^*} = G \cap \mathbb{H}_{\Omega \setminus \{z^*\}}$), $m \in \omega$ and $(p, H) \in \mathbb{H}_{\Omega \setminus \{z^*\}}$, then $\mathrm{dom}(p) = \{z^*\} \times n_p$. Extend (p, H) to $(q, K) \in \mathbb{H}_{\Omega \setminus \{z^*\}}$ as follows: let $n \geq \max\{n_p, m\}$, $n_q = h(n, H)$, $\mathrm{dom}(p) = \{z^*\} \times n_q$, $K = H$ and $q \restriction \{z^*\} \times [n_p, n) = 0$. In addition, given $i \in [n, n_q)$ put $q(z^*, i) = 1$ if and only if $i \notin \bigcup_{w \in H} A_w$. Thus $(q, K) \Vdash [n, h(n, H)) \setminus \bigcup_{w \in H} A_w \subseteq A_{z^*}$ as we wanted. $\square$

Corollary 3.24. Let γ be an ordinal of uncountable cofinality and let $\langle M_\alpha \rangle_{\alpha \leq \gamma}$ be an increasing sequence of transitive ZFC models such that $[\omega]^{\aleph_0} \cap M_\gamma = \bigcup_{\alpha < \gamma} [\omega]^{\aleph_0} \cap M_\alpha$. Assume that $\mathcal{A} = \{A_\alpha : \alpha < \gamma\} \in M_\gamma$ is a family of infinite subsets of ω such that, for any $\alpha < \gamma$, $\mathcal{A}{\restriction}\alpha \in M_\alpha$ and $A_\alpha \in M_{\alpha+1}$ diagonalizes M_α outside $A{\restriction}\alpha$. Then, for any $x \in [\omega]^{\aleph_0} \cap M_\gamma$, there exists an $\alpha < \gamma$ such that $|x \cap A_\alpha| = \aleph_0$. If additionally, $\mathcal{A}$ is almost disjoint, then $\mathcal{A}$ is mad in M_γ.

The previous corollary implies that the almost disjoint family added by $\mathbb{H}_\Omega$ for Ω uncountable is actually maximal (since $\mathbb{H}_\Omega \cong \mathbb{H}_\gamma$ for some ordinal γ of uncountable cofinality).

Brendle and Fischer used the results above for instance, to find a model where $\mathfrak{b} = \mathfrak{a} = \kappa < \mathfrak{s} = \lambda$. They constructed a matrix iteration in which the first column of the matrix corresponds to the iteration adding a Hechler style mad family $\mathcal{A} = \{A_\alpha : \alpha < \kappa\}$ and this happens to be the mad family witnessing $\mathfrak{a} \leq \kappa$. Note that it lives already in the extension given by the poset $\mathbb{P}_{0,\kappa}$.

This family satisfies the hypothesis of Corollary 3.24 and each A_α will be preserved to diagonalize the models in the α-th row outside $\mathcal{A}{\restriction}\alpha$. Now we mention the preservation results obtained by Brendle-Fischer and additionally prove that the class of forcings that satisfies the diagonalization property is actually broader.

Lemma 3.25 ([BF11, Lemma 11])**.** Let $\mathbb{P} \in M$ be a poset. If $N \models$ "a^* diagonalizes M outside A" then

$$N^{\mathbb{P}} \models \text{"}a^* \text{ diagonalizes } M^{\mathbb{P}} \text{ outside } A.\text{"}$$

Proof. The proof of this lemma is analogous to the one in Lemma 3.11 (1). □

Corollary 3.26. If $N \models$ "a^* diagonalizes M outside A" then

$$N^{\mathbb{C}^N} \models \text{"}a^* \text{ diagonalizes } M^{\mathbb{C}^M} \text{ outside } A.\text{"}$$

The proof of the Lemma below is an argument similar to the one Miller used to prove that after forcing with $\mathbb{E}$, the ground model reals remain unbounded in the generic extension. This argument is sometimes called a *compactness argument*.

Lemma 3.27. If $N \models$ "a^* diagonalizes M outside A" then

$$N^{\mathbb{E}^N} \models \text{"}a^* \text{ diagonalizes } M^{\mathbb{E}^M} \text{ outside } \mathcal{A}.\text{"}$$

Proof. Let $\dot{h} \in M$ be an $\mathbb{E}$-name for a function from $\omega \times [\Omega]^{<\omega}$ into ω. Work within M and fix a non-principal ultrafilter $\mathcal{D}$ on ω (in M). For $s \in \omega^{<\omega}$ and $n < \omega$ define $h_{s,n} : \omega \times [\Omega]^{<\omega} \to \omega + 1$ as

$$h_{s,n}(i, F) = \begin{cases} \min\{j < \omega : (\forall \varphi,\ \mathrm{width}(\varphi) \leq n)((s, \varphi) \nVdash \dot{h}(i, F) > j)\} & \text{if it exists} \\ \omega & \text{otherwise} \end{cases}$$

Claim 3.28. $h_{s,n}(i, F) \in \omega$ for all $i < \omega$ and $F \in [\Omega]^{<\omega}$.

Proof. Assume not, so there is a sequence of slaloms $(\varphi_j : j < \omega)$ of width $\leq n$ such that $(s, \varphi_j) \Vdash \dot{h}(i, F) > j$. Define the slalom φ^* as

$$\varphi^*(i) = \{m < \omega : \{j < \omega : m \in \varphi_j(i)\} \in \mathcal{D}\}.$$

Note that width$(\varphi^*) \leq n$: if not, there exists $i \in \omega$ such that $|\varphi^*(i)| > n$, let $\{m_0, \dots, m_l\}$ be an enumeration of $\varphi^*(i)$ with $l > n$, hence we have that for all $k \leq l$, the set $X_l = \{j < \omega : m_l \in \varphi_j(i)\} \in \mathcal{D}$ and so does the set $X = \bigcap_{k \leq l} X_l$. But then, for all indexes $j \in X$, $|\varphi_j(i)| > n$ which is a contradiction. This implies that (s, φ^*) is indeed a condition in $\mathbb{E}$.

Take $(t, \psi) \leq (s, \varphi^*)$ and $j_0 < \omega$ such that $(t, \psi) \Vdash \dot{h}(i, F) = j_0$. By the definition of φ^*, $\{j < \omega : \forall i \in |t| \setminus |s| (t(i) \notin \varphi_j(i))\} \in \mathcal{D}$, so in particular infinite. For any $j > j_0$ in that set, (t, ψ) is compatible with (s, φ_j) and, therefore, any common stronger condition forces $j_0 = \dot{h}(i, F) > j$, a contradiction. $\square$

We finish now the proof of the lemma: In N fix $m < \omega$ and $p = (s, \varphi) \in \mathbb{E}^N$ with $n :=$ width(φ). Since a^* diagonalizes M outside $\mathcal{A}$, there are $i \geq m$ and $F \in [\Omega]^{<\omega}$ such that $[i, h_{s,n}(i, F)) \setminus \bigcup_{z \in F} A_z \subseteq a^*$. By definition of $h_{s,n}$, $(\forall \varphi,$ width$(\varphi) \leq n)((s, \varphi) \nVdash \dot{h}(i, F) > h_{s,n}(i, F))$ is a true $\mathbf{\Pi}^1_1$-statement in M and so, by absoluteness, it is also true in N. Therefore, there is a $q \in \mathbb{E}^N$ stronger than p that forces $\dot{h}(i, F) \leq h_{s,n}(i, F)$ and then we conclude that q forces $[i, \dot{h}(i, F)) \setminus \bigcup_{z \in F} A_z \subseteq a^*$. $\square$

Finally, the following shows that we have the preservation property of mad families when forcing with random algebras:

Lemma 3.29. If $N \models$ "a^* diagonalizes M outside $\mathcal{A}$" then:

$$N^{\mathbb{B}^N} \models \text{"}a^* \text{ diagonalizes } M^{\mathbb{B}^M} \text{ outside } \mathcal{A}\text{"}.$$

Proof. In the standard proof that $\mathbb{B}$ is ω^ω-bounding (see for example [BJ95]) it is shown that, for any $p \in \mathbb{B}$, $\epsilon \in (0, 1)$ and $\dot{x}$ a $\mathbb{B}$-name for a real in ω^ω, there are $q \leq p$ and $g \in \omega^\omega$ such that $q \Vdash \dot{x} \leq g$ and $\lambda(p \setminus q) \leq \epsilon\lambda(p)$ where λ is the Lebesgue measure. We are going to use this fact to prove the lemma.

Fix $\dot{h} \in M$ a $\mathbb{B}$-name for a function from $\omega \times [\Omega]^{<\omega}$ to ω, $p \in \mathbb{B}^N$ and $m < \omega$. By the Lebesgue density theorem there is a clopen non-empty set C such that $\lambda(C \setminus p) < \frac{1}{4}\lambda(C)$. Now, in M using the property mentioned above find $g : \omega \times [\Omega]^{<\omega} \to \omega$ such that, for any $F \in [\Omega]^{<\omega}$, there is a $q_F \leq C$ in $\mathbb{B}$ with $\lambda(C \setminus q_F) \leq \frac{1}{4}\lambda(C)$ that forces $\forall i < \omega(\dot{h}(i, F) \leq g(i, F))$.

Thus, in N there are $i \geq m$ and $F \in [\Omega]^{<\omega}$ such that $[i, g(i, F)) \setminus \bigcup_{z \in F} a_z \subseteq a^*$, so q_F forces $[i, \dot{h}(i, F)) \setminus \bigcup_{z \in F} a_z \subseteq a^*$. Seeing that $\lambda(p \cap q_F) > \frac{1}{2}\mu(C)$ we have $p \cap q_F \in \mathbb{B}^N$ is stronger than p and forces $[i, \dot{h}(i, F)) \setminus \bigcup_{z \in F} a_z \subseteq a^*$. $\square$

Corollary 3.30. Let $\Gamma \in M$ be a non-empty set. If $N \models$ "a^* diagonalizes M outside $\mathcal{A}$" then:

$$N^{\mathbb{B}^N_\Gamma} \models \text{"}a^* \text{ diagonalizes } M^{\mathbb{B}^M_\Gamma} \text{ outside } \mathcal{A}.\text{"}$$

The following lemma ensures that the diagonalization property is preserved in limit steps of a finite support iteration of ccc posets. The proof is analogous to the one in Lemma 3.11 (2), so we omit it

Lemma 3.31 ([BF11, Lemma 12]). Let s be a coherent pair of finite support iterations, $\mathcal{A} \in V$ a family of infinite subsets of ω and $\dot{a}^*$ a $\mathbb{P}_{i_1,0}$-name for an infinite subset of ω such that:

$$\Vdash_{\mathbb{P}_{i_1,\xi}} \text{“}\dot{a}^* \text{ diagonalizes} V_{i_0,\xi} \text{ outside } \mathcal{A}\text{”}$$

for all $\xi < \pi$. Then, $\mathbb{P}_{0,\pi} \lessdot \mathbb{P}_{1,\pi}$ and $\Vdash_{\mathbb{P}_{1,\pi}}$ “$\dot{a}^*$ diagonalizes $V_{0,\pi}$ outside $\mathcal{A}$”.

Proof. The proof of this lemma is analogous to the one in Lemma 3.11 (2). □

The results above are summarized in the following theorem for the case of 2D-coherent systems.

Theorem 3.32. *Let* $\mathbf{m}$ *be a standard 2D-coherent system with* $I^{\mathbf{m}} = \gamma + 1$ *and ordinal and* $\pi^{\mathbf{m}} = \pi$ *satisfying (i) and (ii) of Lemma 3.16 and, for each* $\beta < \gamma$*, let* $\dot{a}_\alpha$ *be a* $\mathbb{P}_{\alpha+1,0}$*-name of an infinite subset of* ω *such that* $\mathbb{P}_{\alpha+1,0}$ *forces that* $\dot{A}_\alpha$ *diagonalizes* $V_{\alpha,0}$ *outside* $\{\dot{A}_\varepsilon : \varepsilon < \alpha\}$ *and* $\mathbb{P}_{\gamma,0}$ *forces* $\dot{\mathcal{A}} = \{\dot{A}_\alpha : \alpha < \gamma\}$ *to be an almost disjoint family. If* $\mathbb{S}_\xi \in \{\mathbb{C}, \mathbb{E}\} \cup \mathfrak{R}$ *for all* $\xi \in S$ *then* $\mathbb{P}_{\gamma,\pi}$ *forces that* $\dot{\mathcal{A}}$ *is mad and* $\mathfrak{a} \leq |\gamma|$.

Remark 3.33. A version of the previous theorem was originally proved by Brendle and Fischer [BF11] for a special case where Mathias-Příkrý posets are considered and the associated ultrafilters are built carefully. Namely, given $(M_\alpha : \alpha \leq \gamma)$ and an almost disjoint family $\mathcal{A}$ as in Corollary 3.24 and $M_0 \models$“$\mathcal{U}_0$ is an ultrafilter on ω”, Brendle and Fischer constructed an increasing chain of filters $(\mathcal{U}_\alpha : \alpha \leq \gamma) \in M_\gamma$ such that, for all $\beta \leq \gamma$:

1. $(\mathcal{U}_\alpha : \alpha \leq \beta) \in M_\beta$ and $\mathcal{U}_\beta$ is an ultrafilter in M_β.
2. $\mathbb{M}(\mathcal{U}_\alpha) \lessdot_{M_\alpha} \mathbb{M}(\mathcal{U}_\beta)$ for all $\alpha < \beta$.
3. If $\beta = \alpha + 1$ then $\mathbb{M}(\mathcal{U}_\beta)$ forces that a_α diagonalizes $M_\alpha^{\mathbb{M}(\mathcal{U}_\alpha)}$ outside $\mathcal{A}{\restriction}\alpha$.

The idea of this construction is originated from a similar method by Blass and Shelah [BS89] to preserve unbounded reals. Instead of an almost disjoint family $\mathcal{A}$, they consider a sequence of reals $(c_\alpha : \alpha < \gamma) \in M_\gamma$ such that each $c_\alpha \in M_{\alpha+1}$ is unbounded over M_α and the ultrafilters are constructed such that (1) and (2) above are satisfied and $\mathbb{M}(\mathcal{U}_{\alpha+1})$ forces that c_α is unbounded over $M_\alpha^{\mathbb{M}(\mathcal{U}_\alpha)}$ for all $\alpha < \gamma$.

The following is a generalization of a result of Steprāns [Ste93] which shows that the maximal almost disjoint family added by the forcing $\mathbb{H}_\kappa$ is indestructible after forcing with some particular posets. Steprāns' result can be then deduced when $\kappa = \omega_1$ (so $\mathbb{H}_{\omega_1} = \mathbb{C}_{\omega_1}$) and $\dot{\mathbb{Q}}_\xi = \mathbb{C}$ for all $\xi < \pi$.

Theorem 3.34. *Let κ be an uncountable regular cardinal. After forcing with $\mathbb{H}_\kappa$, any finite support iteration $(\mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi : \xi < \pi)$ where each iterand is either*

1. *in $\{\mathbb{C}, \mathbb{E}\} \cup \mathfrak{R}$ or*
2. *a ccc poset of size $< \kappa$*

preserves the mad family added by $\mathbb{H}_\kappa$.

Proof. We reconstruct the iteration $\mathbb{H}_\kappa$ followed by $(\mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi : \xi < \pi)$ as a standard 2D-coherent system $\mathbf{m}$ so that $\mathbb{P}^{\mathbf{m}}_{\kappa,\xi} = \mathbb{P}_\xi$ for all $\xi \leq \pi$. The construction goes as follows (see Definition 3.18):

1. $I^{\mathbf{m}} = \kappa + 1$ and $\pi^{\mathbf{m}} = \pi$.
2. For each $\alpha \leq \kappa$, $\mathbb{P}^{\mathbf{m}}_{\alpha,0} = \mathbb{H}_\alpha$.
3. The partition $(S^{\mathbf{m}}, C^{\mathbf{m}})$ of $\pi^{\mathbf{m}}$ corresponds to the set of ordinals in the iteration where a poset coming from (1) or (2) is used. In other words, $\xi \in S^{\mathbf{m}}$ if $\Vdash_{\mathbb{P}_\xi} \dot{\mathbb{Q}}_\xi \in \{\mathbb{C}, \mathbb{E}\} \cup \mathfrak{R}$, and $\xi \in C^{\mathbf{m}}$ otherwise.
4. The functions $\Delta^{\mathbf{m}} : C^{\mathbf{m}} \to \kappa$ and the sequences $(\mathbb{S}^{\mathbf{m}}_\xi : \xi \in S^{\mathbf{m}})$ and $(\dot{\mathbb{Q}}^{\mathbf{m}}_\xi : \xi \in C^{\mathbf{m}})$ are constructed by recursion on $\xi < \pi$ along with the finite support iterations of the 2D-coherent system. We split into the following cases:
   - If $\xi \in S^{\mathbf{m}}$ define $\mathbb{S}^{\mathbf{m}}_\xi$ to be one of the posets in the set $\{\mathbb{C}, \mathbb{E}\} \cup \mathfrak{R}$ depending on what $\mathbb{P}_\xi$ forces $\dot{\mathbb{Q}}_\xi$ to be.
   - If $\xi \in C^{\mathbf{m}}$ we define both $\Delta^{\mathbf{m}}(\xi)$ and $\dot{\mathbb{Q}}^{\mathbf{m}}_\xi$, the latter as a $\mathbb{P}^{\mathbf{m}}_{\Delta^{\mathbf{m}}(\xi),0}$-name. Since $\xi \in C^{\mathbf{m}}$ we have that $\dot{\mathbb{Q}}_\xi$ is a $\mathbb{P}^{\mathbf{m}}_{\kappa,\xi}$-name for a ccc poset of size $< \kappa$, hence without loss of generality we can assume that the domain of $\dot{\mathbb{Q}}_\xi$ is an ordinal $\gamma_\xi < \kappa$ (not just a name). By Lemma 3.16, $\dot{\mathbb{Q}}_\xi$ is (forced by $\mathbb{P}^{\mathbf{m}}_{\kappa,\xi}$ to be equal to) a $\mathbb{P}^{\mathbf{m}}_{\alpha,\xi}$-name $\dot{\mathbb{Q}}^{\mathbf{m}}_\xi$ for some $\alpha < \kappa$. So put $\Delta^{\mathbf{m}}(\xi) = \alpha + 1$.

Notice that $\mathbf{m}$ satisfies the assumptions of Theorem 3.32 for the mad family A added by $\mathbb{H}_\kappa$, so A is still mad in $V^{\mathbf{m}}_{\kappa,\pi}$.

□

Remark 3.35. When $\kappa = \omega_1$ in Theorem 3.34 the result still holds when $\mathbb{H}_{\omega_1}$ is replaced by any finite support iteration of length with cofinality ω_1. This is a generalization of Zhang's result [Zha99] which states that, under CH, there is a mad family in the ground model which stays mad after a finite support iteration of $\mathbb{E}$.

3.4. Consistency results on Cichoń's diagram

In this section, we prove the consistency of certain constellations in Cichoń's diagram where additionally, the almost disjointness number can be decided (equal to $\mathfrak{b}$). For all the results, we fix uncountable regular cardinals $\theta_0 \leq \theta_1 \leq \kappa \leq \mu \leq \nu$ and a cardinal $\lambda \geq \nu$. We denote the ordinal product between cardinals by, e.g., $\lambda \cdot \mu$.

The following summarizes the results in [Mej13, Sect. 3] but in addition we get that $\mathfrak{b} = \mathfrak{a}$ can be forced.

Theorem 3.36. *Assume $\lambda = \lambda^{<\kappa}$ and $\lambda' \geq \lambda$ with $(\lambda')^{\aleph_0} = \lambda'$. For each of the items below, there is a ccc poset forcing the corresponding statement.*

1. $\mathrm{add}(\mathcal{N}) = \theta_0$, $\mathrm{cov}(\mathcal{N}) = \theta_1$, $\mathfrak{b} = \mathfrak{a} = \mathrm{non}(\mathcal{M}) = \kappa$ *and* $\mathrm{cov}(\mathcal{M}) = \mathfrak{c} = \lambda$.
2. $\mathrm{add}(\mathcal{N}) = \theta_0$, $\mathrm{cov}(\mathcal{N}) = \theta_1$, $\mathfrak{b} = \mathfrak{a} = \kappa$, $\mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \mu$ *and* $\mathfrak{d} = \mathrm{non}(\mathcal{N}) = \mathfrak{c} = \lambda$.
3. $\mathrm{add}(\mathcal{N}) = \theta_0$, $\mathfrak{b} = \mathfrak{a} = \kappa$, $\mathrm{cov}(\mathcal{I}) = \mathrm{non}(\mathcal{I}) = \mu$ *for* $\mathcal{I} \in \{\mathcal{M}, \mathcal{N}\}$ *and* $\mathfrak{d} = \mathfrak{c} = \lambda$.
4. $\mathrm{non}(\mathcal{N}) = \aleph_1$, $\mathfrak{b} = \mathfrak{a} = \kappa$, $\mathfrak{d} = \lambda$ *and* $\mathrm{cov}(\mathcal{N}) = \mathfrak{c} = \lambda'$.

Proof. The proofs are basically the same as in [Mej13] combined with the methods of preservation of mad families developed in Section 3.3 which we include in this paper for completeness. For all the items, start adding a mad family with $\mathbb{H}_\kappa$.

1. Construct an iteration as in the last part of [Mej13, Theorem 2]. To be more precise, perform a finite support iteration $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \lambda)$ where each $\dot{\mathbb{Q}}_\alpha$ is either:
   a) a σ-linked subposet of $\mathbb{LOC}$ of size $<\theta_0$,
   b) a subalgebra of $\mathbb{B}$ of size $<\theta_1$ or
   c) a σ-centered subposet of $\mathbb{D}$ of size $<\kappa$.

   To be more specific, in a) $\dot{\mathbb{Q}}_\alpha$ is of the form $\mathbb{LOC}^\alpha$ where this is a $\mathbb{P}_\alpha$-name of σ-linked subposet of the localization forcing $\mathbb{LOC}$ in the $\mathbb{P}_\alpha$ extension of size $<\theta_0$.

   By a book-keeping device, the iteration satisfies for every $\alpha < \lambda$:

   a') if $\dot{K}$ is a $\mathbb{P}_\alpha$-name of a subset of ω^ω of size $<\theta_0$, then there is an $\alpha' \in [\alpha, \lambda)$ such that $\dot{\mathbb{Q}}_{\alpha'}$ is as in a) and the slalom it adds localizes all the reals in $\dot{K}$,

   b') if $\dot{B}$ is a $\mathbb{P}_\alpha$-name of a family of size $<\theta_1$ of Borel-null sets (coded in $V^{\mathbb{H}_\kappa * \mathbb{P}_\alpha}$) then there is an $\alpha' \in [\alpha, \lambda)$ such that $\dot{\mathbb{Q}}_{\alpha'}$ is as in b) and the random real it adds is outside the Borel sets in $\dot{B}$ and

   c') if $\dot{F}$ is a $\mathbb{P}_\alpha$-name of a subset of ω^ω of size $<\kappa$, then there is an $\alpha' \in [\alpha, \lambda)$ such that $\dot{\mathbb{Q}}_{\alpha'}$ is as in c) and the generic real it adds dominates the reals in $\dot{F}$.

   For instance, in a') considering the subalgebra $\dot{\mathbb{Q}}_{\alpha'} = \mathbb{LOC}^{\alpha'}$ as above, α' can be found such that $\dot{K}$ is already in the α'-th extension, and so the generic slalom added by this subalgebra localizes all the reals in $\dot{K}$. For b') and c') one argues similarly.

   In order to finish the proof, we present the arguments that show why each cardinal characteristic takes the desired value in the generic extension given by $\mathbb{P}_\lambda$.

   $\underline{\mathrm{add}(\mathcal{N}) = \theta_0}$: The inequality $\mathrm{add}(\mathcal{N}) \leq \theta_0$ follows from both the fact that $\mathrm{add}(\mathcal{N}) = \mathfrak{b}(\mathrm{LOC})$ (see Example 0.24 (4)) and that all the posets we are using are θ_0-LOC-good, so Theorem 0.26 applies and we get $\mathfrak{b}(\mathrm{LOC}) \leq \theta_0$. On the other hand, (a') implies $\mathrm{add}(\mathcal{N}) \geq \theta_0$.

$\underline{\mathrm{cov}(\mathcal{N}) = \theta_1}$: For $\mathrm{cov}(\mathcal{N}) \leq \theta_1$ note that $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{b}(\mathcal{E}_b) \leq \theta_1$ (see Example 0.24 (3)). Thus, since our posets are θ_1-$\mathcal{E}_b$-good, Theorem 0.26 still applies. Conversely, the inequality $\mathrm{cov}(\mathcal{N}) \geq \theta_1$ follows from b').

$\underline{\mathfrak{b} = \mathfrak{a} = \mathrm{non}(\mathcal{M}) = \kappa}$. It is enough to show $\kappa \leq \mathfrak{b}$, $\mathrm{non}(\mathcal{M}) \leq \kappa$ and $\mathfrak{a} \leq \kappa$. For the latter note that the mad family added at the beginning by the forcing $\mathbb{H}_\kappa$ will stay mad after the iteration thanks to Theorem 3.34. Item (c') implies $\mathfrak{b} \geq \kappa$ and $\mathrm{non}(\mathcal{M}) \leq \kappa$ follows from both $\mathrm{non}(\mathcal{M}) = \mathfrak{b}(\mathcal{E})$ and the fact that our posets are κ-$\mathcal{E}$-good.

$\underline{\mathrm{cov}(\mathcal{M}) = \mathfrak{c} = \lambda}$. The inequality $\mathrm{cov}(\mathcal{M}) \geq \lambda$ is a simple consequence from the equality $\mathrm{cov}(\mathcal{M}) = \mathfrak{d}(\mathcal{E})$ together with Theorem 0.26; on the other hand, $\mathfrak{c} \leq \lambda$ because, in the ground model, $|\mathbb{H}_\kappa * \mathbb{P}_\lambda| \leq \lambda$.

2. Like in (1), perform a finite support iteration $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \lambda \cdot \mu)$ as in [Mej13, Theorem 3] where each $\dot{\mathbb{Q}}_\alpha$ is either
   a) a σ-linked subposet of $\mathbb{LOC}$ of size $<\theta_0$,
   b) a subalgebra of $\mathbb{B}$ of size $<\theta_1$,
   c) a σ-centered subposet of $\mathbb{D}$ of size $<\kappa$ or
   d) $\mathbb{E}$.

   By counting arguments, the finite support iteration is constructed such that, for any $\alpha < \mu$,
   (a') if $\dot{K}$ is a $\mathbb{P}_{\lambda\cdot\alpha}$-name of a subset of ω^ω of size $<\theta_0$, then there is a $\xi < \lambda$ such that $\dot{\mathbb{Q}}_{\lambda\cdot\alpha+\xi}$ is as in (i) and the slalom it adds localizes all the reals in $\dot{K}$,
   (b') if $\dot{B}$ is a $\mathbb{P}_{\lambda\cdot\alpha}$-name of a family of size $<\theta_1$ of Borel-null sets (coded in $V^{\mathbb{H}_\kappa * \mathbb{P}_{\lambda\cdot\alpha}}$) then there is a $\xi < \lambda$ such that $\dot{\mathbb{Q}}_{\lambda\cdot\alpha+\xi}$ is as in (ii) and the random real it adds is outside the Borel sets in $\dot{B}$ and
   (c') if $\dot{F}$ is a $\mathbb{P}_{\lambda\cdot\alpha}$-name of a subset of ω^ω of size $<\kappa$, then there is a $\xi < \lambda$ such that $\dot{\mathbb{Q}}_{\lambda\cdot\alpha+\xi}$ is as in (iii) and the generic real it adds dominates the reals in $\dot{F}$.

   The arguments for $\mathrm{add}(\mathcal{N}) = \theta_0$, $\mathrm{cov}(\mathcal{N}) = \theta_1$ and $\mathfrak{b} = \mathfrak{a} = \kappa$ are similar to the ones in (a). We present here the remainder ones.

   $\underline{\mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \mu}$. Both inequalities $\mathrm{cov}(\mathcal{M}) \leq \mu$ and $\mathrm{non}(\mathcal{M}) \geq \mu$ are witnessed by the cofinal μ-many eventually different reals added by $\mathbb{E}$ (note that the cofinality of the iteration is μ). On the other hand, $\mathrm{non}(\mathcal{M}) \leq \mu \leq \mathrm{cov}(\mathcal{M})$ follows by the cofinal μ-many Cohen reals added at limit stages.

   $\underline{\mathfrak{d} = \mathrm{non}(\mathcal{N}) = \mathfrak{c} = \lambda}$. Follows from Theorem 0.26, just recall that $\mathrm{non}(\mathcal{N}) \geq \mathfrak{d}(\mathcal{E}\lceil \restriction b)$.

3. Perform a finite support iteration $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \lambda \cdot \mu)$ as in [Mej13, Theorem 3]. In this case, each $\dot{\mathbb{Q}}_\alpha$ is either:
   a) a σ-linked subposet of $\mathbb{LOC}$ of size $<\theta_0$,
   b) a σ-centered subposet of $\mathbb{D}$ of size $<\kappa$ or
   c) $\mathbb{B}$.

As in (2), the iteration is build up (using counting arguments) so that (a′) and (b′) from the previous proof hold.

a) if $\dot{K}$ is a $\mathbb{P}_{\lambda\cdot\alpha}$-name of a subset of ω^ω of size $<\theta_0$, then there is a $\xi<\lambda$ such that $\dot{\mathbb{Q}}_{\lambda\cdot\alpha+\xi}$ is as in (i) and the slalom that it adds localizes all the reals in $\dot{K}$ and

b) if $\dot{F}$ is a $\mathbb{P}_{\lambda\cdot\alpha}$-name of a subset of ω^ω of size $<\kappa$, then there is a $\xi<\lambda$ such that $\dot{\mathbb{Q}}_{\lambda\cdot\alpha+\xi}$ is as in (iii) and the generic real that it adds dominates the reals in $\dot{F}$.

The arguments for $\mathrm{add}(\mathcal{N})=\theta_0$, $\mathfrak{b}=\mathfrak{a}=\kappa$, $\mathrm{cov}(\mathcal{M})=\mathrm{non}(\mathcal{M})=\mu$ and $\mathfrak{d}=\mathfrak{c}=\lambda$ are similar to the ones explained in the two items above. Both $\mathrm{cov}(\mathcal{N})\geq\mu$ and $\mathrm{non}(\mathcal{N})\leq\mu$ follow from the cofinally μ-many random reals added along the iteration. Finally it is clear that $\mathrm{cov}(\mathcal{N})\leq\mu$ and $\mathrm{non}(\mathcal{N})\geq\mu$.

4. After the iteration in (a) force with $\mathbb{B}_{\lambda'}$. Clearly, the first $\aleph_1$-many random reals added by this algebra form a non-null set of reals in the final extension, so $\mathrm{non}(\mathcal{N})=\aleph_1$. The set of all λ'-many random reals added by the algebra give us a witness for $\mathrm{cov}(\mathcal{N})\geq\lambda'$.

□

3.4.1. Consistency results using 3D-coherent systems

Now we turn to prove some consistency results with standard 3D-coherent systems. Recall that if $\mathbf{t}$ is such a system with $I^{\mathbf{t}}=(\gamma+1)\times(\delta+1)$, standard 2D-coherent systems $\mathbf{t}_\alpha$ can be extracted for each $\alpha\leq\gamma$ and $\mathbf{t}^\beta$ for each $\beta\leq\delta$. The following result will be crucial in the upcoming results.

Theorem 3.37. *Let $\mathbf{t}$ be a standard 3D-coherent system with $I^{\mathbf{t}}=(\gamma+1)\times(\delta+1)$ and $\mathbf{m}$ a standard 2D-coherent system with with $I^{\mathbf{m}}=\gamma+1$ and $\pi^{\mathbf{m}}=\delta$ such that $\mathbb{P}_{\alpha,\beta,0}=\mathbb{P}^{\mathbf{t}}_{\alpha,\beta,0}=\mathbb{P}^{\mathbf{m}}_{\alpha,\beta}$ for all $\alpha\leq\gamma$ and $\beta\leq\delta$. Let $\mathcal{R}=\langle X,Y,\sqsubset\rangle$ be a Polish relational system coded in V. Assume*

1. *$\mathbf{m}$ satisfies the hypotheses of either*
   - *a) Lemma 3.16 (1) and (2) and Theorem 3.19 with $(\dot{c}_\alpha:\alpha<\gamma)$ and $\mathcal{R}$, or*
   - *b) Theorem 3.32 with $\dot{\mathcal{A}}=\{\dot{A}_\alpha:\alpha<\gamma\}$*

   (note that, in either case, δ has uncountable cofinality),
2. *all the posets that form $\mathbf{m}$ are non-trivial (see Definition 3.18(c) and (d)),*
3. *all the posets that form $\mathbf{t}$ are non-trivial (see Definition 3.18(c) and (d)),*
4. *γ and π have uncountable cofinality,*
5. *for $\xi\in S=S^{\mathbf{t}}$, $\dot{\mathbb{Q}}_{\alpha,\beta,\xi}$ is forced to be $\mathcal{R}$-good by $\mathbb{P}_{\alpha,\beta,\xi}$ for all $\alpha\leq\gamma$ and $\beta\leq\delta$, and*
6. *if (1)(b) is assumed then $\mathbb{S}_\xi\in\{\mathbb{C},\mathbb{E}\}\cup\mathfrak{R}$ for all $\xi\in S$.*

Then, $\mathbb{P}_{\gamma,\delta,\pi}$ forces

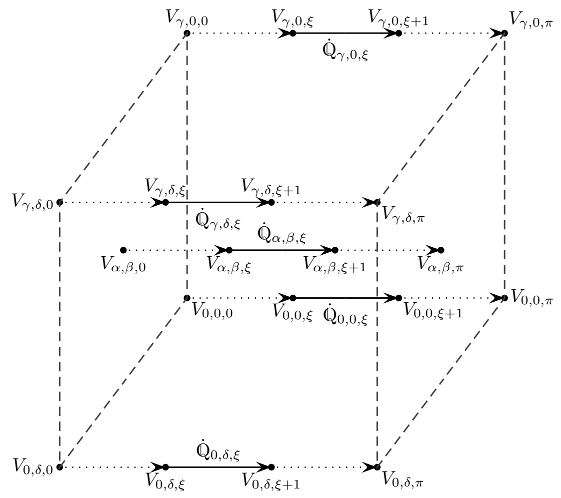

Figure 3.2.: Cube of generic extensions (3D-coherent system).

(a') $\mathrm{non}(\mathcal{M}) \leq \mathrm{cf}(\pi) \leq \mathrm{cov}(\mathcal{M})$,

(b') $\mathfrak{b}_{\sqsubset} \leq \min\{\mathrm{cf}(\delta), \mathrm{cf}(\pi)\} \leq \max\{\mathrm{cf}(\delta), \mathrm{cf}(\pi)\} \leq \mathfrak{d}_{\sqsubset}$,

(c') $\mathfrak{b}_{\sqsubset} \leq \min\{\mathrm{cf}(\gamma), \mathrm{cf}(\delta), \mathrm{cf}(\pi)\} \leq \max\{\mathrm{cf}(\gamma), \mathrm{cf}(\delta), \mathrm{cf}(\pi)\} \leq \mathfrak{d}_{\sqsubset}$ *when (1)(a) is assumed and*

(d') $\mathfrak{a} \leq |\delta|$ *when (1)(b) is assumed.*

Proof.

Any finite support iteration of length π of uncountable cofinality adds cofinal $\mathrm{cf}(\pi)$-many Cohen reals which witness $\mathrm{non}(\mathcal{M}) \leq \mathrm{cf}(\pi) \leq \mathrm{cov}(\mathcal{M})$. Also note that the finite support iteration $(\mathbb{P}_{\gamma,\delta,\xi}, \dot{\mathbb{Q}}_{\gamma,\delta,\xi} : \xi < \pi)$ generates the final extension $V_{\gamma,\delta,\pi}$ of the coherent system $\mathbf{t}$.

2. We look at the 2D-coherent system $\mathbf{t}_\gamma$. As the chain of posets $(\mathbb{P}_{\gamma,\beta,0} : \beta \leq \delta)$ is generated by a finite support iteration of ccc posets, for a fixed cofinal sequence $(\beta_\zeta : \zeta < \mathrm{cf}(\delta))$ in δ of limit ordinals, for each $\zeta < \mathrm{cf}(\delta)$ there is a $\mathbb{P}_{\gamma,\beta_{\zeta+1},0}$-name $\dot{c}'_\zeta$ for a Cohen real over $V_{\gamma,\beta_\zeta,0}$. Thus, $\mathbf{t}_\gamma$ and $(\dot{c}'_\zeta : \zeta < \mathrm{cf}(\delta))$ satisfy the hypotheses of Theorem 3.19 by (5.), so $\mathbb{P}_{\gamma,\delta,\pi}$ forces $\mathfrak{b}(\mathcal{R}) \leq \mathrm{cf}(\delta) \leq \mathfrak{d}(\mathcal{R})$. Besides, since $\mathfrak{b}(\mathcal{R}) \leq \mathrm{non}(\mathcal{M})$ and $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{d}(\mathcal{R})$, (a') immediately implies $\mathfrak{b}(\mathcal{R}) \leq \mathrm{cf}(\pi) \leq \mathfrak{d}(\mathcal{R})$.

3. We first look at the 2D-coherent system $\mathbf{m}$. By Theorem 3.19, $\mathbb{P}_{\alpha+1,\delta,0}$ forces that $\dot{c}_\alpha$ is $\mathcal{R}$-unbounded over $V_{\alpha,\delta,0}$ for every $\alpha < \gamma$. Now, we apply Theorem 3.19 to $\mathbf{t}^\delta$ to conclude that $\mathfrak{b}(\mathcal{R}) \leq \mathrm{cf}(\gamma) \leq \mathfrak{d}(\mathcal{R})$.

4. By Theorem 3.32 applied to the 2D-coherent system $\mathbf{m}$, each $\dot{A}_\alpha$ is forced by $\mathbb{P}_{\alpha+1,\delta,0}$ to diagonalize $V_{\alpha,\delta,0}$ outside $\dot{\mathcal{A}}{\restriction}\alpha$ for each $\alpha < \gamma$ and furthermore, using

the same theorem one more time for the coherent system $\mathbf{t}^\delta$, $\mathbb{P}_{\alpha+1,\delta,\pi}$ forces that $\dot{A}_\alpha$ diagonalizes $V_{\alpha,\delta,\pi}$ outside $\dot{\mathcal{A}}{\restriction}\alpha$. Thus the maximality of $\mathcal{A}$ is preserved in $V_{\gamma,\delta,\pi}$ and so $\mathfrak{a} \leq |\gamma|$.

□

In our applications and in accordance with the previous result, we consider standard 3D-coherent systems where $(\mathbb{P}_{\alpha,\beta,0} : \alpha \leq \gamma, \beta \leq \delta)$ is generated by a standard 2D-coherent system.

Definition 3.38. Given ordinals γ and δ, define the following standard 2D-coherent systems.

1. The system $\mathbf{m}^{\mathbb{C}}(\gamma,\delta)$ where
   a) $I^{\mathbf{m}^{\mathbb{C}}(\gamma,\delta)} = \gamma + 1$,
   b) $\mathbb{P}_{\alpha,0}^{\mathbf{m}^{\mathbb{C}}(\gamma,\delta)} = \mathrm{Fn}(\alpha \times \omega, 2)$ for each $\alpha \leq \gamma$ (recall that $\mathrm{Fn}(\Omega, 2)$ is the poset of finite partial functions from Ω to 2 ordered by reverse inclusion), and
   c) $\pi^{\mathbf{m}^{\mathbb{C}}(\gamma,\delta)} = \delta$, $S = \delta$, $C = \emptyset$ and $\mathbb{S}_\beta = \mathbb{C}$ for all $\beta < \delta$.
2. The system $\mathbf{m}^*(\gamma,\delta)$ where
   a) $I^{\mathbf{m}^*(\gamma,\delta)} = \gamma + 1$,
   b) $\mathbb{P}_{\alpha,0}^{\mathbf{m}^*(\gamma,\delta)} = \mathbb{H}_\alpha$ for each $\alpha \leq \gamma$, and
   c) $\pi^{\mathbf{m}^*(\gamma,\delta)} = \delta$, $S = \delta$, $C = \emptyset$ and $\mathbb{S}_\beta = \mathbb{C}$ for all $\beta < \delta$.

If both γ and δ have uncountable cofinality, it is clear that both $\mathbf{m}^{\mathbb{C}}(\gamma,\delta)$ and $\mathbf{m}^*(\gamma,\delta)$ satisfy (1) and (2) of theorem above, moreover, the former satisfies (1)(a) and the latter satisfies (1)(b). These standard 2D-coherent systems are the starting point for the 3D-coherent systems constructed to prove the main results below. Observe that this results below are "three-dimensional versions" of the 2D-coherent systems constructed in [Mej13, Section 6].

We first prove that there is a constellation of Cichoń's diagram with 7 different values as illustrated in Figure 3.3.

Theorem 3.39 (Main Result). *Assume $\lambda^{<\theta_1} = \lambda$. Then, there is a ccc poset forcing* $\mathrm{add}(\mathcal{N}) = \theta_0$, $\mathrm{cov}(\mathcal{N}) = \theta_1$, $\mathfrak{b} = \mathfrak{a} = \kappa$, $\mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \mu$, $\mathfrak{d} = \nu$ *and* $\mathrm{non}(\mathcal{N}) = \mathfrak{c} = \lambda$.

Proof. Let $V = V_{0,0,0}$ be the ground model where we perform a finite support iteration which comes from the standard 3D-coherent system $\mathbf{t}$ constructed as follows. Fix a bijection $g = \langle g_0, g_1, g_2 \rangle : \lambda \to \kappa \times \nu \times \lambda$.

1. $\gamma = \kappa + 1$, $\delta = \nu + 1$ and $\pi = \lambda \cdot \nu \cdot \mu$.
2. $\langle \mathbb{P}_{\alpha,\beta,0} : \alpha \leq \kappa, \beta \leq \nu \rangle$ is obtained from $\mathbf{m}^*(\kappa,\nu)$.
3. Consider $\lambda \cdot \nu \cdot \mu$ as the disjoint union of the $\nu \cdot \mu$-many intervals $I_\zeta = [l_\zeta, l_{\zeta+1})$ (for $\zeta < \nu \cdot \mu$) of order type λ. Let $S := \{l_\zeta : \zeta < \nu \cdot \mu\}$ and $C = \pi \smallsetminus S$ (note that $l_\zeta = \lambda \cdot \zeta$).

4. A function $\Delta = \langle \Delta_0, \Delta_1 \rangle : C \to \kappa \times \nu$ such that the following properties are satisfied:
   a) For all $\xi < \pi$, both $\Delta_0(\xi)$ and $\Delta_1(\xi)$ are successor ordinals,[2]
   b) $\Delta^{-1}(\alpha+1, \beta+1) \cap \{l_\zeta + 1 : \zeta < \nu \cdot \mu\}$ is cofinal in π for any $(\alpha, \beta) \in \kappa \times \nu$, and
   c) for fixed $\zeta < \nu \cdot \mu$ and $e < 2$, $\Delta(l_\zeta + 2 + 2 \cdot \varepsilon + e) = (g_0(\varepsilon) + 1, g_1(\varepsilon) + 1)$ for all $\varepsilon < \lambda$.
5. $\mathbb{S}_\xi = \mathbb{E}$ for all $\xi \in S$.
6. Fix, for each $\alpha < \kappa$, $\beta < \nu$ and $\zeta < \nu \cdot \mu$, two sequences $\langle \dot{\mathbb{LOC}}^{\zeta}_{\alpha,\beta,\eta} \rangle_{\eta<\lambda}$ and $\langle \dot{\mathbb{B}}^{\zeta}_{\alpha,\beta,\eta} \rangle_{\eta<\lambda}$ of $\mathbb{P}_{\alpha,\beta,l_\zeta}$-names for all σ-linked subposets of the localization forcing $\mathbb{LOC}^{V_{\alpha,\beta,l_\zeta}}$ of size $<\theta_0$ and all subalgebras of random forcing $\mathbb{B}^{V_{\alpha,\beta,l_\zeta}}$ of size $<\theta_1$, respectively[3].

   Given $\xi \in C$, define $\dot{\mathbb{Q}}_\xi$ according to the following cases.
   a) If $\xi = l_\zeta + 1$ then $\dot{\mathbb{Q}}_\xi$ is a $\mathbb{P}_{\Delta(\xi),\xi}$-name for the poset $\mathbb{D}^{V_{\Delta(\xi),\xi}}$, the Hechler poset adding a dominating real $\dot{d}_\zeta$ over the model $V_{\Delta(\xi),\xi}$.
   b) If $\xi = l_\zeta + 2 + 2\varepsilon$ then $\dot{\mathbb{Q}}_\xi = \dot{\mathbb{LOC}}^{\zeta}_{g(\varepsilon)}$.
   c) If $\xi = l_\zeta + 2 + 2\varepsilon + 1$ then $\dot{\mathbb{Q}}_\xi = \dot{\mathbb{B}}^{\zeta}_{g(\varepsilon)}$.

We prove that $V_{\kappa,\nu,\pi}$ satisfies the statements of this theorem.

Claim 3.40. If $X \in V_{\kappa,\nu,\pi}$ is a set of reals of size $<\mu$, then there are $(\beta, \zeta) \in \nu \times (\nu \cdot \mu)$ so that $X \in V_{\kappa,\beta,l_\zeta}$. Furthermore, if $|X| < \kappa$, then there is also an $\alpha < \kappa$ such that $X \in V_{\alpha,\beta,l_\zeta}$.

Proof. As $\mathrm{cf}(\pi) = \mu$ and $V_{\kappa,\nu,\pi}$ is obtained by a finite support iteration of length π, there is a $\zeta < \nu \cdot \mu$ such that $X \in V_{\kappa,\nu,l_\zeta}$ (because $\{l_\zeta : \zeta < \nu \cdot \mu\}$ is cofinal in π). Now, look at the 2D-coherent system $\mathbf{t}_\kappa$ and apply Corollary 3.17 to find a $\beta < \nu$ so that $X \in V_{\kappa,\beta,l_\zeta}$. In the case that $|X| < \kappa$, apply Corollary 3.17 to $\mathbf{t}^\beta$ to find an $\alpha < \kappa$ so that X belongs to V_{α,β,l_ζ}. □

<u>$\mathrm{add}(\mathcal{N}) = \theta_0$</u>. For the inequality $\mathrm{add}(\mathcal{N}) \geq \theta_0$ take an arbitrary set X of reals in $V_{\kappa,\nu,\pi}$ of size $<\theta_0$ so, by Claim 3.40, there is a triple of ordinals $(\alpha, \beta, \zeta) \in \kappa \times \nu \times (\nu \cdot \mu)$ such that $X \in V_{\alpha,\beta,l_\zeta}$. In V_{α,β,l_ζ}, there is a transitive model N of (a large enough finite fragment of) ZFC such that $X \subseteq N$ and $|N| < \theta_0$. Then, there exists an $\eta < \lambda$ such that $\mathbb{LOC}^{\zeta}_{\alpha,\beta,\eta} = \mathbb{LOC}^N$. Put $\varepsilon = g^{-1}(\alpha, \beta, \eta)$ and $\xi' = l_\zeta + 2 + 2\varepsilon$, so $\mathbb{Q}_{\xi'} = \mathbb{LOC}^{\zeta}_{\alpha,\beta,\eta} = \mathbb{LOC}^N$ adds a generic slalom over N and, therefore, it localizes all the reals in X.

[2]Both ordinals $\Delta_0(\xi)$ and $\Delta_1(\xi)$ are successor because, if they are limits of uncountable cofinality and we force with $\mathbb{D}^V_{\Delta(\xi),\xi}$ above $(\Delta(\xi), \xi)$ and trivial otherwise, then $\mathbb{R} \cap V_{\Delta(\xi),\xi+1}$ may not be $\mathbb{R} \cap \bigcup_{\alpha<\Delta_0(\xi),\beta<\Delta_1(\xi)} V_{\alpha,\beta,\xi+1}$.

[3]Instead of localization, we could enumerate all the ccc posets from V_{α,β,l_ζ} of size $<\theta_0$ that are ccc in $V_{\gamma,\delta,l_\zeta}$ to force, in the end, $\mathrm{MA}_{<\theta_0}$.

To obtain the converse inequality, apply Theorem 0.26 to $(\mathbb{P}_{\kappa,\nu,\xi}, \dot{\mathbb{Q}} : \xi < \pi)$.

$\underline{\mathrm{cov}(\mathcal{N}) = \theta_1}$. This case is similar to the one above. To get $\mathrm{cov}(\mathcal{N}) \geq \theta_1$ take an arbitrary family Z of Borel null sets coded in $V_{\kappa,\nu,\pi}$ of size $< \theta_1$ so, by Claim 3.40, there exists $(\alpha, \beta, \zeta) \in \kappa \times \nu \times (\nu \cdot \mu)$ such that the sets in Z are already coded in V_{α,β,l_ζ}. Hence, as in the previous argument, there exists an ordinal $\eta < \lambda$ such that the generic random real added by $\mathbb{B}^{\zeta}_{\alpha,\beta,\eta}$ avoids all the Borel sets in Z. Put $\varepsilon = g^{-1}(\alpha, \beta, \xi)$ and $\zeta' = l_\zeta + 2 + 2\varepsilon + 1$, so $\mathbb{Q}_{\zeta'} = \mathbb{B}^{\zeta}_{\alpha,\beta,\eta}$ and the random real it adds is already in $V_{\alpha+1,\beta+1,\zeta'+1}$.

Conversely, since the posets we use in the finite support iteration $(\mathbb{P}_{\kappa,\nu,\xi}, \dot{\mathbb{Q}}_{\kappa,\nu,\xi} : \xi < \pi)$ are θ_1-$\mathcal{E}_b$-good posets and $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{b}(\mathcal{E}{\restriction_b})$, Theorem 0.26 implies that, in $V_{\kappa,\nu,\pi}$, $\mathfrak{b}(\mathcal{E}{\restriction_b}) \leq \theta_1$.

$\underline{\mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \mu}$. The inequalities $\mathrm{non}(\mathcal{M}) \leq \mu \leq \mathrm{cov}(\mathcal{M})$ follow from Theorem 3.37(a). Conversely, from the cofinal μ-many eventually different reals added by the iteration $(\mathbb{P}_{\kappa,\nu,\xi}, \dot{\mathbb{Q}}_{\kappa,\nu,\xi} : \xi < \pi)$, we force the inequalities $\mathrm{cov}(\mathcal{M}) \leq \mu$ and $\mathrm{non}(\mathcal{M}) \geq \mu$.

$\underline{\mathrm{add}(\mathcal{M}) = \mathfrak{b} = \mathfrak{a} = \kappa}$. Given a family F of reals in $V_{\kappa,\nu,\pi}$ of size $<\kappa$, we can find a $(\alpha, \beta, \zeta) \in \kappa \times \nu \times (\nu \cdot \mu)$ such that $F \in V_{\alpha,\beta,l_\zeta}$. We use now the restricted dominating reals $\{\dot{d}_\zeta : \zeta < \nu \cdot \mu\}$. Since $(\Delta)^{-1}(\alpha+1, \beta+1) \cap \{l_\zeta + 1 : \zeta < \nu \cdot \mu\}$ is cofinal in $\nu \cdot \mu$, there exists a $\zeta' \in [\zeta, \nu \cdot \mu)$ such that $\Delta(l_{\zeta'} + 1) = (\alpha + 1, \beta + 1)$ and then the real $\dot{d}_{\zeta'}$ added by $\mathbb{Q}_{\alpha+1,\beta+1,\xi'}$, where $\xi' = l_{\zeta'} + 1$, dominates all the reals in F.

On the other hand, $\mathfrak{a} \leq \kappa$ follows from Theorem 3.37 which guarantees that the mad family added along the α-axis, which lives in the model $V_{\kappa,0,0}$, still remains mad in the final extension $V_{\kappa,\nu,\pi}$.

$\underline{\mathfrak{d} = \mathrm{cof}(\mathcal{M}) = \nu}$. For $V_{\kappa,\nu,\pi} \models \mathfrak{d} \geq \nu$ we just use Theorem 3.37. Conversely, to see $V^{\mathbb{P}} \models \mathfrak{d} \leq \nu$ note that the argument above shows that the family of (restricted) dominating reals $\{\dot{d}_\zeta : \zeta < \nu \cdot \mu\}$ is dominating in $V_{\kappa,\nu,\pi}$.

$\underline{\mathrm{non}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \mathfrak{c} = \lambda}$. As $\mathfrak{d}(\mathcal{E}{\restriction_b}) \leq \mathrm{non}(\mathcal{N})$, from Theorem 0.26 we have that, in $V_{\kappa,\nu,\pi}$, $\mathfrak{d}(\mathcal{E}{\restriction_b}) \geq |\pi| = \lambda$. Certainly, $\mathfrak{c} \leq \lambda$ holds because $|\mathbb{P}_{\kappa,\nu,\pi}| = \lambda$.

$\square$

Once we have obtained such a constellation, we can do some modifications to obtain different ones, where ther cardinal invariants are separated.

Theorem 3.41. *Assume $\lambda^{<\theta_0} = \lambda$. Then, for any of the statements below, there is a ccc poset forcing it.*

1. $\mathrm{add}(\mathcal{N}) = \theta_0$, $\mathfrak{b} = \mathfrak{a} = \kappa$, $\mathrm{cov}(\mathcal{I}) = \mathrm{non}(\mathcal{I}) = \mu$ *for* $\mathcal{I} \in \{\mathcal{M}, \mathcal{N}\}$, $\mathfrak{d} = \nu$ *and* $\mathrm{cof}(\mathcal{N}) = \mathfrak{c} = \lambda$.
2. $\mathrm{add}(\mathcal{N}) = \theta_0$, $\mathrm{cov}(\mathcal{N}) = \kappa$, $\mathrm{add}(\mathcal{M}) = \mathrm{cof}(\mathcal{M}) = \mu$, $\mathrm{non}(\mathcal{N}) = \nu$ *and* $\mathrm{cof}(\mathcal{N}) = \mathfrak{c} = \lambda$.
3. $\mathrm{add}(\mathcal{N}) = \theta_0$, $\mathrm{cov}(\mathcal{N}) = \mathfrak{b} = \mathfrak{a} = \kappa$, $\mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \mu$, $\mathfrak{d} = \mathrm{non}(\mathcal{N}) = \nu$ *and* $\mathrm{cof}(\mathcal{N}) = \mathfrak{c} = \lambda$.

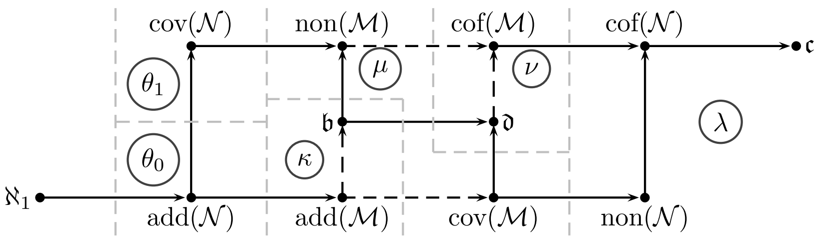

Figure 3.3.: Cichoń's diagram as in Theorem 3.39.

Proof. Fix bijection $g : \lambda \to \kappa \times \nu \times \lambda$. All the 3D-coherent systems we use in this proof are of the form $\mathbf{t}$ where

1. $\gamma = \kappa + 1$, $\delta = \nu + 1$ and $\pi = \lambda \cdot \nu \cdot \mu$, the latter of which is the disjoint union of $\nu \cdot \mu$-many intervals $\{I_\zeta := [l_\zeta, l_{\zeta+1}) : \zeta < \nu \cdot \mu\}$ of length λ where each $l_\zeta := \lambda \cdot \zeta$.
2. $S = \{l_\zeta : \zeta < \nu \cdot \mu\}$ and $C = \pi \smallsetminus S$.
3. For (a) and (c) $(\mathbb{P}_{\alpha,\beta,0} : \alpha \leq \kappa, \beta \leq \nu)$ comes from $\mathbf{m}^*(\kappa, \nu)$ and, for (b), it comes from from $\mathbf{m}^{\mathbb{C}}(\kappa, \nu)$.
4. A function $\Delta = (\Delta_0, \Delta_1) : C \to \kappa \times \nu$ such that the following properties are satisfied:
   a) For all $\xi < \pi$, both $\Delta_0(\xi)$ and $\Delta_1(\xi)$ are successor ordinals,
   b) $\Delta^{-1}(\alpha + 1, \beta + 1) \cap \{l_\eta + 1 : \eta < \nu \cdot \mu\}$ is cofinal in π for each $(\alpha, \beta) \in \kappa \times \nu$; additionally, for (c), $\Delta^{-1}(\alpha + 1, \beta + 1) \cap \{l_\eta + 2 : \eta < \nu \cdot \mu\}$ is cofinal in π and
   c) for fixed $\zeta < \nu \cdot \mu$, $\Delta(l_\zeta + n_0 + \varepsilon) = (g_0(\varepsilon) + 1, g_1(\varepsilon) + 1)$ for all $\varepsilon < \lambda$, where $n_0 = 2$ for (a) and (b), and $n_0 = 3$ for (c).

For each of the item below, $\mathbf{t}$ is defined appropriately.

1. For all $\xi \in S$, $\mathbb{S}_\xi = \mathbb{B}$. Fix, for each $\alpha < \kappa$, $\beta < \nu$ and $\zeta < \nu \cdot \mu$, a sequence $(\dot{\mathbb{LOC}}^{\zeta}_{\alpha,\beta,\eta} : \eta < \lambda)$ of $\mathbb{P}_{\alpha,\beta,l_\zeta}$-names for all σ-linked subposets of $\mathbb{LOC}^{V_{\alpha,\beta,l_\zeta}}$ of size $<\theta_0$. For $\xi \in C$, $\dot{\mathbb{Q}}_\xi$ is defined according to the following cases.
   a) If $\xi = l_\zeta + 1$ then $\dot{\mathbb{Q}}_\xi$ is a $\mathbb{P}_{\Delta(\xi),\xi}$-name for the poset $\mathbb{D}^{V_{\Delta(\xi),\xi}}$ which adds a dominating real $\dot{d}_\zeta$ over $V_{\Delta(\xi),\xi}$.
   b) If $\xi = l_\zeta + n_0 + \varepsilon$ for some $\varepsilon < \lambda$, then $\dot{\mathbb{Q}}_\xi = \dot{\mathbb{LOC}}^{\zeta}_{g(\varepsilon)}$.

   Most of the arguments for each of the cardinals characteristics are identical to the ones presented in Theorem 3.39, so we just present the missing ones.

   <u>$\mathrm{non}(\mathcal{N}) \leq \mu \leq \mathrm{cov}(\mathcal{N})$.</u> It holds because we add cofinal μ-many random reals (corresponding to the coordinates $\xi \in S$).

   <u>$\mathrm{cof}(\mathcal{N}) \geq \lambda$.</u> It is a consequence of both the fact that $\mathrm{cof}(\mathcal{N}) = \mathfrak{d}(\mathbb{LOC})$ and Theorem 0.26 which gives us $\mathfrak{d}(\mathbb{LOC}) \geq |\pi| = \lambda$.

2. For all $\xi \in S$, $\mathbb{S}_\xi = \mathbb{D}$ and, for $\xi \in C$, $\dot{\mathbb{Q}}_\xi$ is defined as in (a) but, in (i), we consider $\mathbb{B}^{V_{\Delta(\xi),\xi}}$ instead.

   Recall that, in this construction, our base 2D-coherent system comes from $\mathbf{m}^{\mathbb{C}}(\kappa, \nu)$. The argument to prove that $V_{\kappa,\nu,\pi}$ satisfy (b) is similar to (a) and to the proof of Theorem 3.39. For instance,

   $\underline{\mathrm{cov}(\mathcal{N}) = \kappa \text{ and } \mathrm{non}(\mathcal{N}) = \nu}$. Given a family X of Borel-null sets coded in $V^{\mathbb{P}}$ of size $<\kappa$, we can find $(\alpha, \beta, \zeta) \in \kappa \times \nu \times (\nu \cdot \mu)$ such that all the sets in X are already coded in V_{α,β,l_ζ}. Since $\Delta^{-1}(\alpha+1, \beta+1) \cap \{l_\zeta + 1 : \zeta < \nu \cdot \mu\}$ is cofinal in $\nu \cdot \mu$, there exists $\zeta' \in [\zeta, \lambda)$ such that $\Delta(l_{\zeta'}+1) = (\alpha+1, \beta+1)$ and then the random real $\dot{r}_{\zeta'}$ added by $\dot{\mathbb{Q}}_{\alpha,\beta,\xi'}$ with $\xi' = l_{\zeta'}+1$ avoids all the sets in X. Note that this same argument also proves that the set $\{\dot{r}_\zeta : \zeta < \nu \cdot \mu\}$ is not null, so $\mathrm{non}(\mathcal{N}) \leq \nu$. Conversely, $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{b}(\mathcal{E}\lceil_b) \leq \kappa$ and $\nu \leq \mathfrak{d}(\mathcal{E}\lceil_b) \leq \mathrm{non}(\mathcal{N})$ are direct consequences of Theorem 3.37.

   $\underline{\mathfrak{b} = \mathfrak{d} = \mu}$. Because the cofinal μ-many dominated reals added by the forcing $(\mathbb{P}_{\kappa,\nu,\xi}, \dot{\mathbb{Q}}_{\kappa,\nu,\xi} : \xi < \pi)$ form a scale of length μ.

3. For all $\xi \in S$, $\mathbb{S}_\xi = \mathbb{E}$. For $\xi \in C$, $\dot{\mathbb{Q}}_\xi$ is defined according to the following cases:
   a) If $\xi = l_\zeta + 1$, then $\dot{\mathbb{Q}}_\xi$ is a $\mathbb{P}_{\Delta(\xi),\xi}$-name for the poset $\mathbb{D}^{V_{\Delta(\xi),\xi}}$.
   b) If $\xi = l_\zeta + 2$, then $\dot{\mathbb{Q}}_\xi$ is a $\mathbb{P}_{\Delta(\xi),\xi}$-name for the poset $\mathbb{B}^{V_{\Delta(\xi),\xi}}$.
   c) Otherwise, like (ii) of the proof of (a).

$\square$

Theorem 3.42. *Assume $\lambda^{\aleph_0} = \lambda$. Then, for any of the statements below, there is a ccc poset forcing it.*

1. $\mathrm{add}(\mathcal{N}) = \mathrm{cov}(\mathcal{N}) = \mathfrak{b} = \mathfrak{a} = \kappa$, $\mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \mu$, $\mathfrak{d} = \mathrm{non}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \nu$ *and* $\mathfrak{c} = \lambda$.
2. $\mathrm{add}(\mathcal{N}) = \mathfrak{b} = \mathfrak{a} = \kappa$, $\mathrm{cov}(\mathcal{I}) = \mathrm{non}(\mathcal{I}) = \mu$ *for* $\mathcal{I} \in \{\mathcal{M}, \mathcal{N}\}$, $\mathfrak{d} = \mathrm{cof}(\mathcal{N}) = \nu$ *and* $\mathfrak{c} = \lambda$.
3. $\mathrm{add}(\mathcal{N}) = \mathrm{cov}(\mathcal{N}) = \kappa$, $\mathrm{add}(\mathcal{M}) = \mathrm{cof}(\mathcal{M}) = \mu$, $\mathrm{non}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \nu$ *and* $\mathfrak{c} = \lambda$.
4. $\mathrm{add}(\mathcal{N}) = \kappa$, $\mathrm{cov}(\mathcal{N}) = \mathrm{add}(\mathcal{M}) = \mathrm{cof}(\mathcal{M}) = \mathrm{non}(\mathcal{N}) = \mu$, $\mathrm{cof}(\mathcal{N}) = \nu$ *and* $\mathfrak{c} = \lambda$.

Proof. Fix a bijection $g : \lambda \to \kappa \times \nu \times \lambda$. The 3D-coherent systems we use in this proof are of the form $\mathbf{t}$ where

1. $\gamma = \kappa + 1$, $\delta = \nu + 1$ and $\pi = \lambda \cdot \nu \cdot \mu$ is a disjoint union of $\{I_\xi = [l_\zeta, l_{\zeta+1}) : \zeta < \nu \cdot \mu\}$ as in Theorem 3.39.
2. $C = \{l_\zeta : \zeta < \nu \cdot \mu\}$ and $S = \pi \smallsetminus C$.
3. For items (a) and (b) $(\mathbb{P}_{\alpha,\beta,0} : \alpha \leq \kappa, \beta \leq \nu)$ comes from $\mathbf{m}^*(\kappa, \nu)$; for (c) and (d), it comes from $\mathbf{m}^{\mathbb{C}}(\kappa, \nu)$.

4. A function $\Delta = (\Delta_0, \Delta_1) : C \to \kappa \times \nu$ such that the following properties are satisfied:
   a) For all $\xi < \pi$, both $\Delta_0(\xi)$ and $\Delta_1(\xi)$ are successor ordinals and
   b) $\Delta^{-1}(\alpha+1, \beta+1) \cap \{l_\zeta : \zeta < \nu \cdot \mu\}$ is cofinal in π.

1. Put $\mathbb{S}_\xi = \mathbb{E}$ for all $\xi \in S$. For $\xi \in C$, $\dot{\mathbb{Q}}_\xi = \mathbb{LOC}^{V_{\Delta(\xi),\xi}}$.

   We just prove $\text{add}(\mathcal{N}) = \text{cov}(\mathcal{N}) = \mathfrak{b} = \kappa$ and $\mathfrak{d} = \text{non}(\mathcal{N}) = \text{cof}(\mathcal{N}) = \nu$. If X is a set of reals in $V_{\kappa,\nu,\pi}$ of size $<\kappa$, there is a $(\alpha, \beta, \zeta) \in \kappa \times \nu \times (\nu \cdot \mu)$ such that $X \in V_{\alpha,\beta,l_\zeta}$. Since $\Delta^{-1}(\alpha+1, \beta+1) \cap \{l_\zeta : \zeta < \mu\}$ is cofinal in $\nu \cdot \mu$, there exists a $\zeta' \in [\zeta, \lambda)$ such that $\Delta(l_{\zeta'}) = (\alpha+1, \beta+1)$ and then the slalom $\dot{\varphi}_{\zeta'}$ added by $\dot{\mathbb{Q}}_{\alpha,\beta,l_{\zeta'}}$ localizes all the reals in X. Note that $\{\dot{\varphi}_\zeta : \zeta < \nu \cdot \mu\}$ witnesses $\text{cof}(\mathcal{N}) \leq \nu$.

   The inequalities $\mathfrak{b}, \text{cov}(\mathcal{N}) \leq \kappa$ and $\nu \leq \mathfrak{d}, \text{non}(\mathcal{N})$ follow directly from Theorem 3.37.

2. Put $\mathbb{S}_\xi = \mathbb{B}$ for all $\xi \in S$ and, for $\xi \in C$, $\dot{\mathbb{Q}}_\xi$ is as in (a).

   - For $\xi = l_\zeta + 1$, $\dot{\mathbb{Q}}_\xi$ is a $\mathbb{P}_{\Delta(\xi),\xi}$-name for the poset $\mathbb{LOC}^{V_{\Delta(\xi),\xi}}$, the localization forcing adding a slalom $\dot{\varphi}_\xi$ that localizes the reals in the model $V_{\Delta(\xi),\xi}$.
   - For $\xi \neq l_\zeta + 1$, $\dot{\mathbb{Q}}_\xi$ is the trivial forcing.

   All the inequalities can be deduced from previous arguments.

3. Put $\mathbb{S}_\xi = \mathbb{D}$ for all $\xi \in S$ and, for $\xi \in C$, $\dot{\mathbb{Q}}_\xi$ is as in (a)

4. For $\xi \in S$, if it is odd then $\mathbb{S}_\xi = \mathbb{D}$, but when it is even then $\mathbb{S}_{\xi+1} = \mathbb{B}$. For $\xi \in C$, $\dot{\mathbb{Q}}_\xi$ is defined as in (a).

□

We present some other models of constellations of the Cichoń diagram known from [Mej13], where additionally $\mathfrak{b} = \mathfrak{a}$ holds.

Theorem 3.43. *1. If $\lambda^{<\theta_1} = \lambda$ then there is a ccc poset forcing* $\text{add}(\mathcal{N}) = \theta_0$, $\text{cov}(\mathcal{N}) = \theta_1$, $\mathfrak{b} = \mathfrak{a} = \text{non}(\mathcal{M}) = \kappa$, $\text{cov}(\mathcal{M}) = \mathfrak{d} = \nu$ *and* $\text{non}(\mathcal{N}) = \mathfrak{c} = \lambda$.

2. *If $\lambda^{<\theta_0} = \lambda$ then there is a ccc poset forcing* $\text{add}(\mathcal{N}) = \theta_0$, $\text{cov}(\mathcal{N}) = \mathfrak{b} = \mathfrak{a} = \text{non}(\mathcal{M}) = \kappa$, $\text{cov}(\mathcal{M}) = \mathfrak{d} = \text{non}(\mathcal{N}) = \nu$ *and* $\text{cof}(\mathcal{N}) = \mathfrak{c} = \lambda$.

3. *If $\lambda^{\aleph_0} = \lambda$ then there is a ccc poset forcing* $\text{add}(\mathcal{N}) = \text{non}(\mathcal{M}) = \mathfrak{a} = \kappa$, $\text{cov}(\mathcal{M}) = \text{cof}(\mathcal{N}) = \nu$ *and* $\mathfrak{c} = \lambda$.

Proof. This set of models is also obtained from 3D-coherent systems $\mathbf{t}$. Again fix a bijection $g : \lambda \to \kappa \times \nu \times \lambda$, then our systems are defined such that:

- $\gamma = \kappa + 1$, $\delta = \nu + 1$ and the ordinal $\pi = \lambda \cdot \nu \cdot \mu$ will be seen as the disjoint union of $\nu \cdot \mu$-many intervals of length λ, call them $\{I_\xi = [l_\zeta, l_\zeta + \lambda) : \zeta < \nu \cdot \mu\}$.
- $S = \emptyset$ and so $C = \lambda$, this means we do not add full generics.
- The base models $(V_{\alpha,\beta,0} : \alpha < \gamma, \beta < \delta)$ come from $\mathbf{m}^*(\kappa, \nu)$.

- A function $\Delta = \langle \Delta_0, \Delta_1 \rangle : C \to \kappa \times \nu$ such that the following properties are satisfied:
  - For all $\xi < \pi$, both $\Delta_0(\xi)$ and $\Delta_1(\xi)$ are successor ordinals,
  - $\Delta^{-1}(\alpha+1, \beta+1) \cap \{l_\eta + 1 : \eta < \nu \cdot \mu\}$ is cofinal in π and
  - for fixed $\zeta < \nu \cdot \mu$ $\Delta(l_\zeta + 2 + \eta) = (g_0(\eta) + 1, g_1(\eta) + 1)$.

(a) Run the iteration where given $\xi \in C$, $\dot{\mathbb{Q}}_\xi$ is defined according the following rule:

- For $\xi = l_\zeta$, $\dot{\mathbb{Q}}_\xi$ is a $\mathbb{P}_{\Delta(\xi),\xi}$-name for the poset $\mathbb{D}^{V_{\Delta(\xi),\xi}}$, which adds a dominating real over the model $V_{\Delta(\xi),\xi}$.

Now fix, for each $\alpha < \kappa$ and $\beta < \nu$, two sequences $\langle \dot{\mathbb{LOC}}_{\alpha,\beta,\eta} \rangle_{\eta<\lambda}$ and $\langle \dot{\mathbb{B}}_{\alpha,\beta,\eta} \rangle_{\eta<\lambda}$ of $\mathbb{P}_{\alpha,\beta,l_\zeta}$-names for all σ-linked subposets of the localization forcing $\mathbb{LOC}^{V_{\alpha,\beta,l_\zeta}}$ of size $<\theta_0$ and all subalgebras of random forcing $\mathbb{B}^{V_{\alpha,\beta,l_\zeta}}$ of size $<\theta_1$, then:

- If $\xi = l_\zeta + 1 + 2\varepsilon$ for some $\varepsilon < \lambda$, $\dot{\mathbb{Q}}_\xi = \dot{\mathbb{LOC}}_{g(\varepsilon)}$.
- If $\xi = l_\xi + 1 + 2\varepsilon + 1$ for some $\varepsilon < \lambda$, $\dot{\mathbb{Q}}_\xi = \dot{\mathbb{B}}_{g(\varepsilon)}$.

(b) In this case, for $\xi \in C$, $\dot{\mathbb{Q}}_\xi$ is defined according the following rule:

- For $\xi = l_\zeta$, $\dot{\mathbb{Q}}_\xi$ is a $\mathbb{P}_{\Delta(\xi),\xi}$-name for the poset $\mathbb{D}^{V_{\Delta(\xi),\xi}}$, which adds a dominating real over the model $V_{\Delta(\xi),\xi}$.
- For $\xi = l_\zeta + 1$, $\dot{\mathbb{Q}}_\xi$ is a $\mathbb{P}_{\Delta(\xi),\xi}$-name for the poset $\mathbb{B}^{V_{\Delta(\xi),\xi}}$, which adds a random real over the model $V_{\Delta(\xi),\xi}$..

Now fix, for each $\alpha < \kappa$ and $\beta < \nu$, a sequence $\langle \dot{\mathbb{LOC}}_{\alpha,\beta,\eta} \rangle_{\eta<\lambda}$ of $\mathbb{P}_{\alpha,\beta,l_\zeta}$-names for all σ-linked subposets of the localization forcing $\mathbb{LOC}^{V_{\alpha,\beta,l_\zeta}}$ of size $<\theta_0$, then we define:

- If $\xi = l_\zeta + 2 + \varepsilon$ for some $\varepsilon < \lambda$, $\dot{\mathbb{Q}}_\xi = \dot{\mathbb{LOC}}_{g(\varepsilon)}$.

(c) Finally, fix for each $\alpha < \kappa$ and $\beta < \nu$, a sequence $\langle \dot{\mathbb{LOC}}_{\alpha,\beta,\eta} \rangle_{\eta<\lambda}$ of $\mathbb{P}_{\alpha,\beta,l_\zeta}$-names for all σ-linked subposets of the localization forcing $\mathbb{LOC}^{V_{\alpha,\beta,l_\zeta}}$ of size $<\theta_0$, then we define:

- If $\xi = l_\zeta + \varepsilon$ for some $\varepsilon < \lambda$, $\dot{\mathbb{Q}}_\xi = \dot{\mathbb{LOC}}_{g(\varepsilon)}$.

$\square$

3.5. Open questions

1. Assume $\mathbb{P}$ is a Suslin ccc poset coded in M such that $M \models$"$\mathbb{P}$ is $\mathcal{D}$-good" and $N \models$ "a^* diagonalizes M outside $\mathcal{A}$". Does

$$N^{\mathbb{P}^N} \models \text{“}a^* \text{ diagonalizes } M^{\mathbb{S}^M} \text{ outside } \mathcal{A}\text{”?}$$

2. Suppose instead of the almost disjointness number $\mathfrak{a}$ we want to find models where the independence number is decided. Is it possible: (a) to add a canonical independent family in the first step of our coherent system and (b) to have preservation results that guarantee that this independent family will stay maximal?
3. Is it consistent with ZFC (even assuming large cardinals) that (see [BF11]) $\mathfrak{b} < \mathfrak{a} < \mathfrak{s}$?
4. Is it consistent with ZFC that $\mathrm{cov}(\mathcal{M}) < \mathfrak{d} < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N})$? (see [Mej13]).

Chapter 4

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