On Cichoń's Diagram for the uncountable.

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Cardinal Invariants on Cichoń's Diagram

Cichoń's Diagram on the Baire space ω^ω

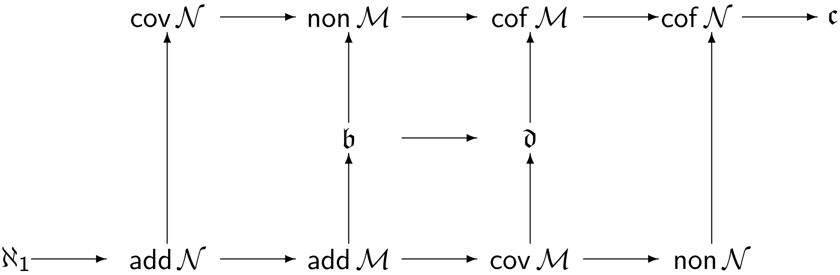


Figure 1: Cichón's diagram

The unbounding and dominating numbers, $\mathfrak{b}(\kappa)$ and $\mathfrak{d}(\kappa)$.

Let κ be a regular cardinal $\geq \omega$.

Definition

If f, g are functions in κ^κ, we say that $f <^* g$, if there exists an $\alpha < \kappa$ such that for all $\beta > \alpha$, $f(\beta) < g(\beta)$. In this case, we say that g eventually dominates f.

Definition

Let $\mathfrak{F}$ be a family of functions from κ to κ.

- *$\mathfrak{F}$ is dominating, if for all $g \in \kappa^\kappa$, there exists an $f \in \mathfrak{F}$ such that $g <^* f$.*
- *$\mathfrak{F}$ is unbounded, if for all $g \in \kappa^\kappa$, there exists an $f \in \mathfrak{F}$ such that $f \not<^* g$.*

Definition

- $\mathfrak{b}(\kappa) = \min\{|\mathfrak{F}|\colon \mathfrak{F}$ *is an unbounded family of functions in* $\kappa^\kappa\}$.
- $\mathfrak{d}(\kappa) = \min\{|\mathfrak{F}|\colon \mathfrak{F}$ *is a dominating family of functions in* $\kappa^\kappa\}$.

Notation: When we refer to the cardinal invariants above in the case $\kappa = \omega$, we will just write $\mathfrak{b}, \mathfrak{d}$.

Cardinal Invariants Associated to an Ideal

Let $\mathcal{I}$ be a σ-ideal on a set X:

Definition

- *The additivity number:*

$$\mathrm{add}(\mathcal{I}) = \min\{|\mathcal{J}|: \mathcal{J} \subseteq \mathcal{I} \textit{ and } \bigcup \mathcal{J} \notin \mathcal{I}\}.$$

- *The covering number:*

$$\mathrm{cov}(\mathcal{I}) = \min\{|\mathcal{J}|: \mathcal{J} \subseteq \mathcal{I} \textit{ and } \bigcup \mathcal{J} = X\}.$$

Definition

- *The cofinality number:*

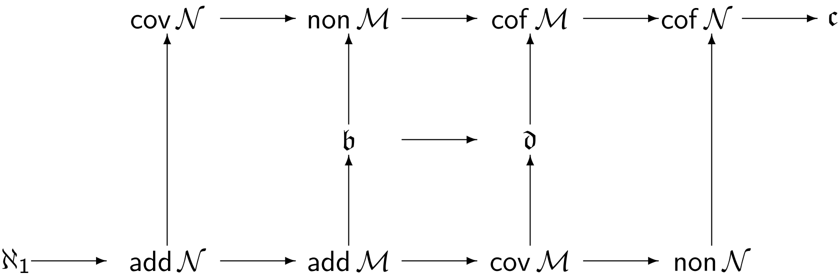
$$\mathrm{cof}(\mathcal{I}) = \min\{|\mathcal{J}| \colon \mathcal{J} \subseteq \mathcal{I} \textit{ and for all } M \in \mathcal{I} \textit{ there is a } J \in \mathcal{J} \textit{ with } M \subseteq J\}.$$

- *The uniformity number:*

$$\mathrm{non}(\mathcal{I}) = \min\{|Y| \colon Y \subset X \textit{ and } Y \notin \mathcal{I}\}.$$

Cichoń's Diagram on the Baire space ω^ω

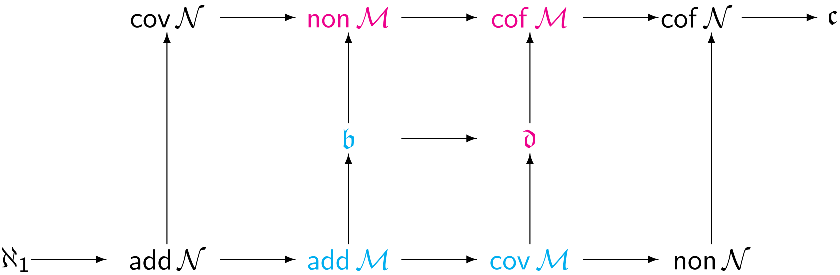
Provable ZFC inequalities between these classical cardinal invariants in the case of the meager and null ideals on ω^ω can be summarized in Cichón's Diagram.



The following relations can also be established in ZFC.

Theorem (Miller and Truss [1].)

$\mathrm{add}\,\mathcal{M} = \min\{\mathfrak{b}, \mathrm{cov}\,\mathcal{M}\}$ *and* $\mathrm{cof}\,\mathcal{M} = \max\{\mathfrak{d}, \mathrm{non}\,\mathcal{M}\}$.



Section 2

The Uncountable Case

Cichoń's Diagram on κ

Let $\kappa > \omega$ be a regular cardinal satisfying $\kappa^{<\kappa} = \kappa$, we are interested in the generalized Baire Space κ^κ and the cardinal invariants associated to it.

As a topological space κ^κ will be endowed with the topology generated by the basic open sets $[s] = \{f \in \kappa^\kappa : \text{and } s \subseteq f\}$ for all $s \in \kappa^{<\kappa}$. Thus we define κ-meager sets to be κ-unions of nowhere dense sets with respect to this topology.

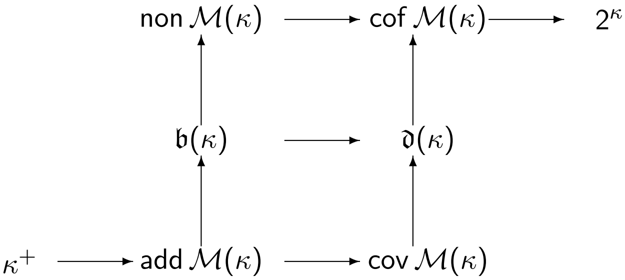


Figure 3: Cichón's Diagram (for κ strongly inaccessible)

Null ideal?

Although there is no suitable notion of measure in the generalized Baire space κ^κ, it is possible to generalize some of the cardinal invariants associated to it (in ω^ω) via their combinatorial characterizations:

Definition

- *A slalom F is a function $F : \kappa \to [\kappa]^{<\kappa}$ such that* $\mathrm{dom}(F) = \kappa$ *and for all $\alpha < \kappa$, $F(\alpha) \in [\kappa]^{|\alpha|}$.*
- *A partial slalom F is a partial map $F : \kappa \to [\kappa]^{<\kappa}$ such that* $\mathrm{dom}(F) \subseteq \kappa$, $|\mathrm{dom}(F)| = \kappa$ *and for all $\alpha \in \mathrm{dom}(F)$,* $F(\alpha) \in [\kappa]^{|\alpha|}$.

More Cardinal Invariants

Definition (Brendle –Brooke-Taylor)

Given a function $f \in \kappa^\kappa$ and a slalom (respectively a partial slalom) F, we say $f \in^* F$ (resp. $f \in^*_p F$) if and only if $\exists\alpha\forall\beta \geq \alpha$ $f(\beta) \in F(\beta)$. (resp. $\exists\alpha < \kappa\forall\beta \geq \alpha$, if $\beta \in \mathrm{dom}(F)$ then $f(\beta) \in F(\beta)$).

Definition

$$\mathfrak{b}(\in^*)(\kappa) = \min\{|\mathcal{F}| \colon \mathcal{F} \subseteq \kappa^\kappa \text{ and } \forall F \text{ slalom } \exists f \in \mathcal{F} \text{ such that } \neg(f \in^* F)\}.$$

$$\mathfrak{d}(\in^*)(\kappa) = \min\{|\mathcal{G}| \colon \mathcal{G} \text{ is a family of slaloms s.t. } \forall f \in \kappa^\kappa\ \exists F \in \mathcal{G}(f \in^* F)\}.$$

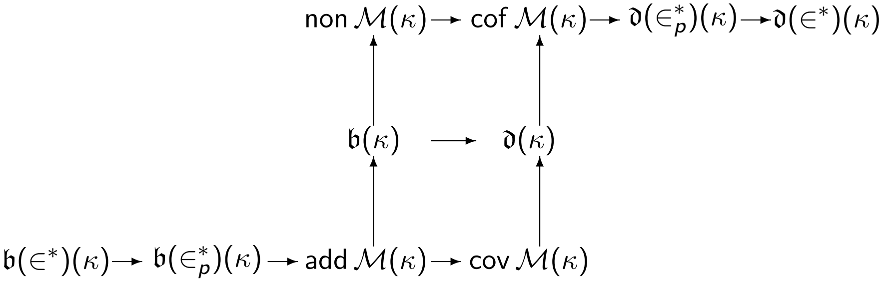


Figure 4: Extended Cichón's Diagram

Section 3

Cardinal Invariants and Forcing

Subsection 1

$<\kappa-$support iterations

κ-Cohen forcing

Let $\mathbb{P}$ the $<\kappa-$product of $\kappa-$Cohen forcing $\mathbb{C}_\kappa = 2^{<\kappa}$ of length $\lambda \geq \kappa^{++}$. Then we have the following properties:

- $\mathbb{C}_\kappa$ has the κ^+-cc and it is κ-closed, and so $\mathbb{P}$ preserves cardinals.
- κ-Cohen functions are dominating over and eventually different from the ground model ones.
- $\mathbb{P}$ preserves unbounded families.

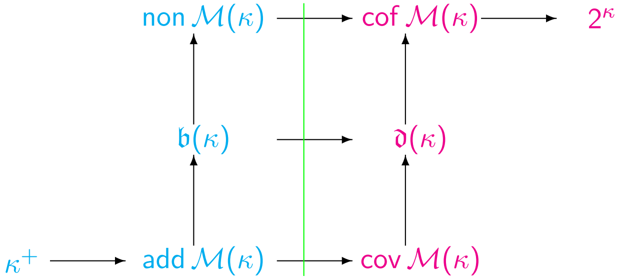


Figure 5: Effect of $\mathbb{P}$ on the diagram

κ-Mathias forcing

In this case we assume κ to be a measurable cardinal and $\mathcal{U}$ to be a normal measure on it. The generalized Mathias Forcing with respect to $\mathcal{U}$ is defined as follows:
$\mathbb{M}^{\kappa}_{\mathcal{U}} = \{(s, A) : s \in [\kappa]^{<\kappa} \text{ and } A \in \mathcal{U}\}$ where $(t, B) \leq (s, A)$ if $t \supseteq s, B \subseteq A$ and $t \setminus s \subseteq A$. It has the following properties:

- $\mathbb{M}^{\kappa}_{\mathcal{U}}$ is κ^{+}-centered and κ-closed.
- $\mathbb{M}^{\kappa}_{\mathcal{U}}$ and $\mathbb{L}^{\kappa}_{\mathcal{U}}$ are forcing equivalent, and as a consequence $\mathbb{M}^{\kappa}_{\mathcal{U}}$ adds dominating functions.

If we iterate $\mathbb{M}^{\kappa}_{\mathcal{U}}$ with $<\kappa$-support and length $\lambda \geq \kappa^{++}$ we obtain $\mathfrak{b}(\kappa) = \lambda = \operatorname{cov}\mathcal{M}(\kappa)$.

κ-Hechler forcing

The generalization of Hechler forcing to κ, $\mathbb{D}_\kappa$ has the form $\mathbb{D}_\kappa = \{(s,f) : s \in \kappa^{<\kappa}$ and $f \in \kappa^\kappa\}$ where $(s,f) \leq (t,g) \leftrightarrow s \supseteq t$, f dominates g everywhere and $\forall\alpha(\mathrm{dom}(t) \leq \alpha < \mathrm{dom}(s) \rightarrow s(\alpha) \geq g(\alpha))$. It has the following properties:

- $\mathbb{D}_\kappa$ is κ^+-centered and κ-closed.
- Generically $\mathbb{D}_\kappa$ adds dominating functions, that also code κ-Cohen functions. If we iterate $\mathbb{D}_\kappa$ with $<\kappa$-support and length $\lambda \geq \kappa^{++}$ we obtain the same effect as with $\mathbb{M}^\kappa_\mathcal{U}$.

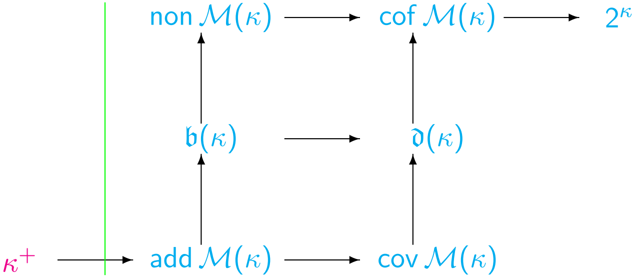


Figure 6: Effect of the iteration with $<\kappa$-support of either $\mathbb{M}_{\mathcal{U}}^{\kappa}$ or $\mathbb{D}_\kappa$

κ-Eventually Different Forcing

The generalization of the eventually different forcing to κ, $\mathbb{E}_\kappa$ has the form: $\mathbb{E}_\kappa = \{(s, F) : s \in \kappa^{<\kappa}$ and $F \in [\kappa^\kappa]^{<\kappa}\}$ where $(s, F) \leq (t, G) \leftrightarrow s \supseteq t$, $F \supseteq G$ and $\forall g \in G$ $\forall \alpha(\mathrm{dom}(t) \leq \alpha < \mathrm{dom}(s) \rightarrow s(\alpha) \neq g(\alpha))$. It has the following properties:

- $\mathbb{E}_\kappa$ is κ^+-centered and κ-closed.
- $\mathbb{E}_\kappa$ adds eventually different functions that will increase non $\mathcal{M}(\kappa)$.

If we iterate $\mathbb{E}_\kappa$ with $<\kappa$-support and length $\lambda \geq \kappa^{++}$ we obtain the following:

- We are adding λ eventually different functions which witness that non $\mathcal{M}(\kappa) = \lambda$.
- For the single step iteration it is possible to preserve the unboundedness of the ground model functions in κ^κ (Using a large cardinal hypothesis on κ).☺
- We don't know if this property can be preserved along the whole iteration.☹

Subsection 2

$\kappa-$support iterations

κ-Sacks Forcing

For strongly inaccessible κ let $\mathbb{S}_\kappa$ be the following forcing notion: Conditions in $\mathbb{S}_\kappa$ are κ-closed subtrees T of $2^{<\kappa}$ such that every node $u \in T$ has a splitting extension in T and the limit of splitting nodes is a splitting node. Also $T \leq S$ if $T \subseteq S$. It satisfies:

- It is possible to define fusion orderings and to define the fusion of a determined sequence of conditions.
- It has the generalized Sacks property, meaning that for every condition $p \in \mathbb{S}_\kappa$ and every $\mathbb{S}_\kappa$-name $\dot{f}$ for an element in κ^κ, there are a condition $q \leq p$ and a slalom $F : \kappa \to [\kappa]^{<\kappa}$ such that $q \Vdash \dot{f}(\alpha) \in F(\alpha)$ for all $\alpha < \kappa$.

If we consider the iteration with κ-support of length κ^{++} we have the following:

- There are also fusion orderings and it is possible to define the fusion of a sequence of conditions in the iteration, so cardinals $\leq \kappa^+$ are preserved.
- It has the generalized Sacks property and, as a consequence $\mathfrak{d}(\in^*)(\kappa)$ as well as the other cardinals in the extended diagram are equal to κ^+.

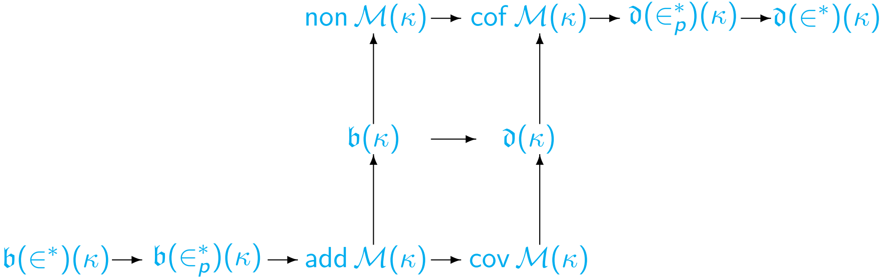


Figure 7: Effect of the iteration of $\kappa-$Sacks forcing

κ-Miller Forcing

Let κ be a measurable cardinal and $\mathcal{U}$ be a κ-complete ultrafilter on it. Define then $\mathbb{MI}^{\kappa}_{\mathcal{U}}$ to be the following forcing notion: Conditions in $\mathbb{MI}^{\kappa}_{\mathcal{U}}$ will be subtrees T of the set of increasing sequences in $\kappa^{<\kappa}$, such that every node can be extended to a $\mathcal{U}$-splitting node (meaning a node with ultrafilter many successors) and the limit of $\mathcal{U}$-splitting nodes is $\mathcal{U}$-splitting.

Note: In order to construct the fusion and to preserve cardinals it is not necessary to consider the ultrafilter version of Miller forcing.

Some properties of this forcing notion are:

- κ-Miller forcing with the club filter $\mathcal{C}$ adds a Cohen subset of κ.
- It is possible to define fusion orderings and to define the fusion of a determined sequence of conditions, and so cardinals $\geq \kappa^+$ are preserved.
- $\mathbb{MI}^\kappa_\mathcal{U}$ generically adds an unbounded function in κ^κ.
- It has the pure decision property meaning that if $T \in \mathbb{MI}^\kappa_\mathcal{U}$, then there is $S \leq T$ with the same stem such that S decides φ i.e. $S \Vdash \varphi$ or $S \Vdash \neg\varphi$.

Our work in progress:

- Does this forcing notion have the generalized Laver property?. Namely, for every condition $p \in \mathbb{P}$, every $g \in V \cap \kappa^\kappa$ and every $\mathbb{P}$-name $\dot{f}$ for an element in κ^κ such that $\Vdash_{\mathbb{P}} \forall \alpha < \kappa(\dot{f}(\alpha) \leq g(\alpha))$ there are a condition $q \leq p$ and a slalom $F : \kappa \to [\kappa]^{<\kappa}$ such that both $|F(\alpha)| \leq 2^{|\alpha|}$ and $q \Vdash \dot{f}(\alpha) \in F(\alpha)$ for all $\alpha < \kappa$.
- What about the iteration of $\mathbb{MI}^\kappa_{\mathcal{U}}$? Has it also the generalized Laver property?

References I

[1] Tomek Bartoszyński and Haim Judah.
Set theory.
A K Peters, Ltd., Wellesley, MA, 1995.
On the structure of the real line.

[2] Andreas Blass.
Combinatorial cardinal characteristics of the continuum.
In *Handbook of set theory. Vols. 1, 2, 3*, pages 395–489. Springer, Dordrecht, 2010.

[3] Andreas Blass, Tapani Hyttinen, and Yi Zhang.
Mad families and their neighbors.
preprint.

[4] James Cummings and Saharon Shelah.
Cardinal invariants above the continuum.
Ann. Pure Appl. Logic, 75(3):251–268, 1995.

[5] Natasha Dobrinen and Sy-David Friedman.
The consistency strength of the tree property at the double successor of a measurable cardinal.
Fund. Math., 208(2):123–153, 2010.

[6] Sy-David Friedman, Radek Honzik, and Lyubomyr Zdomskyy.
Fusion and large cardinal preservation.
Ann. Pure Appl. Logic, 164(12):1247–1273, 2013.

[7] Thomas Jech.
Set Theory. The third millennium edition, revised and expanded.
Springer, 2003.

References II

[8] Akihiro Kanamori.
Perfect-set forcing for uncountable cardinals.
Ann. Math. Logic, 19(1-2):97–114, 1980.

[9] Kenneth Kunen.
Set Theory.
North-Holland, 1980.

[10] Richard Laver.
Making the supercompactness of κ indestructible under κ-directed closed forcing.
Israel Journal of Mathematics, 29(4):385–388, December 1978.